EARTH-LIFE SCIENCE INSTITUTE

Tokyo Institute of Technology

Earth-Life Science C: Planets

Instructors:

- Hidenori Genda
- John Hernlund
- Shigeru Ida
- Hiroyuki Kurokawa

Hernlund

- Lecture 1: The present-day Earth (Tuesday, 4 October)
- Lecture 2: Earth's history (Friday, 7 October)
- Lecture 3: Exploration of the Solar System (Tuesday, 11 October)

Kurokawa

- Lecture 4: Planetary structure and equations (Friday, 14 October)
- Lecture 5: Planetary atmospheres (Tuesday, 18 October)
- Lecture 6: Climate evolution, volatile cycling, and biogeochemical cycling (Friday, 21 October)

Ida

- Lecture 7: Planet formation ~~(Tuesday, 1 November)~~ **Friday, 28 October**
- Lecture 8: Satellite formation (Friday, 4 November)
- Lecture 9: Origins of organic materials (Tuesday, 8 November)

Genda

- Lecture 10: Water delivery to Earth (Friday, 11 November)
- Lecture 11: Stellar evolution (Tuesday, 15 November)
- Lecture 12: Exoplanet observations (Friday, 18 November)

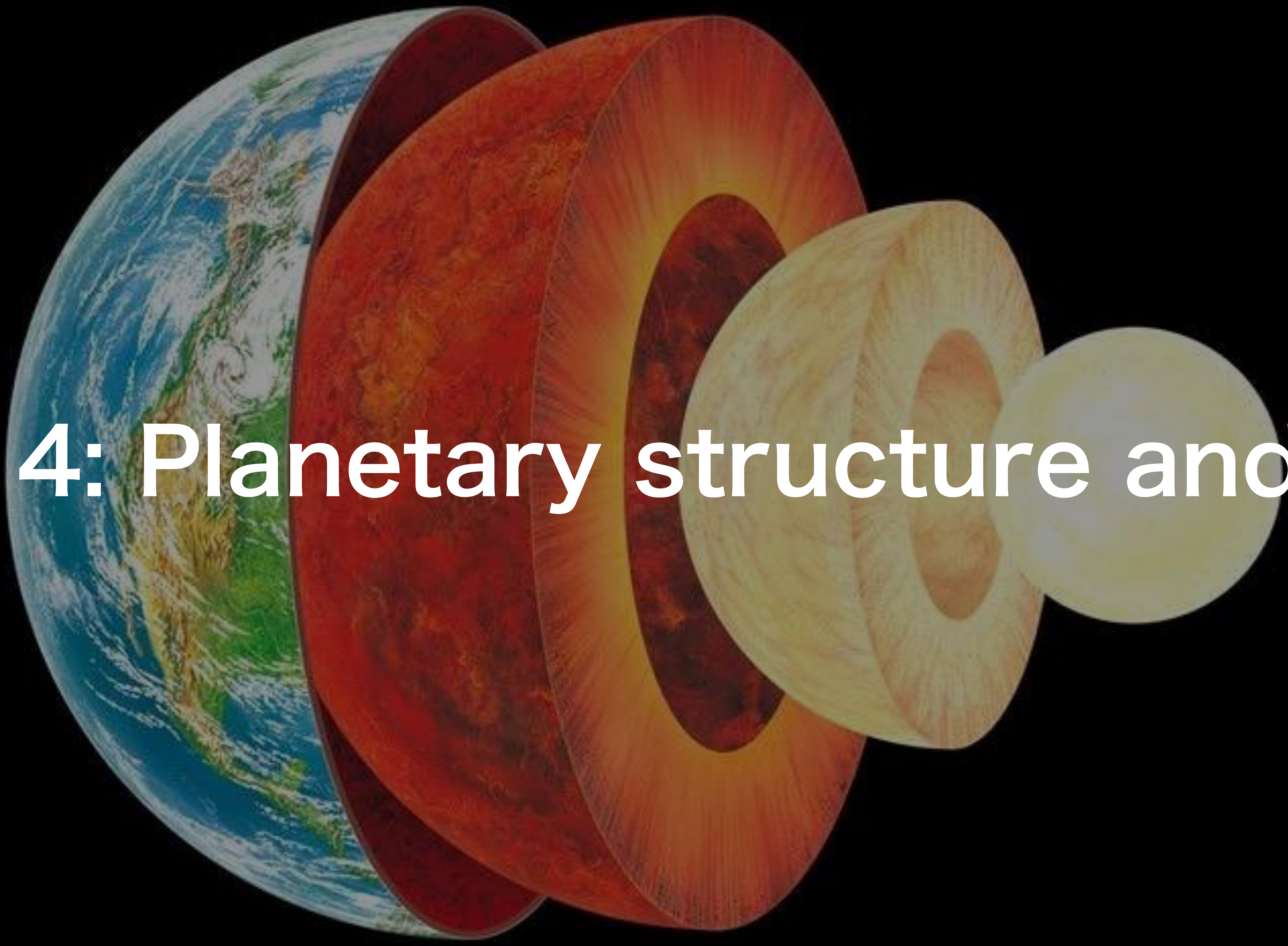
All

- Lecture 13: Summary and future prospects (Tuesday, 22 November)

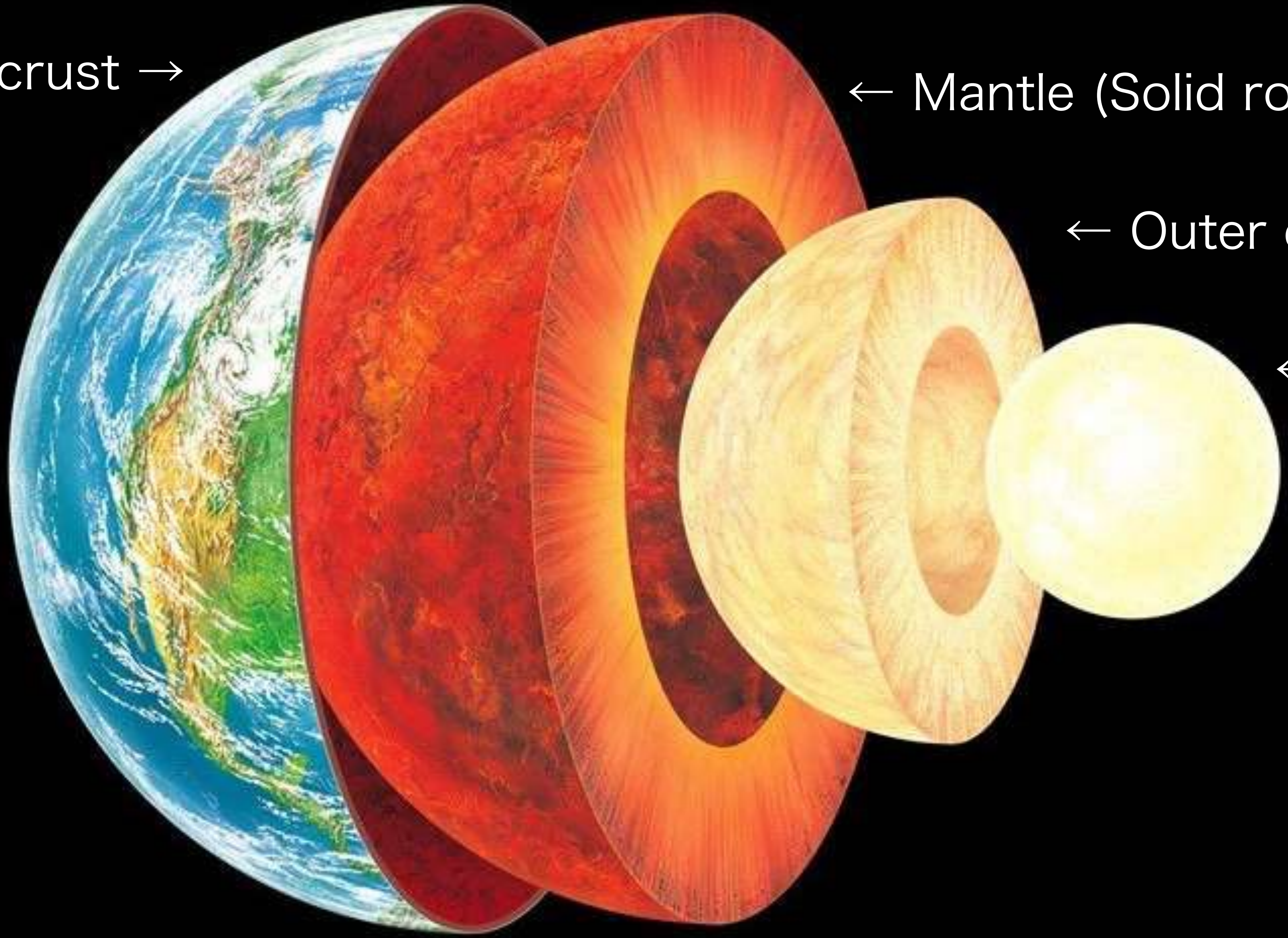
About grade (score)

- Each lecturer will grade on a 25-point scale, and your grade will be the sum of the scores.
- The grade for my lectures will be evaluated based on weekly small reports. I will explain the assignments at the end of each lecture.

Lecture 4: Planetary structure and equations



Structure of Earth’s interior



- Rocky mantle + crust：67.5 wt.%， Metallic core：32.5 wt.%
- Atmosphere：Oceans：Sold Earth = 8×10^{-6}：2×10^{-4}：1

Interiors of rocky planets

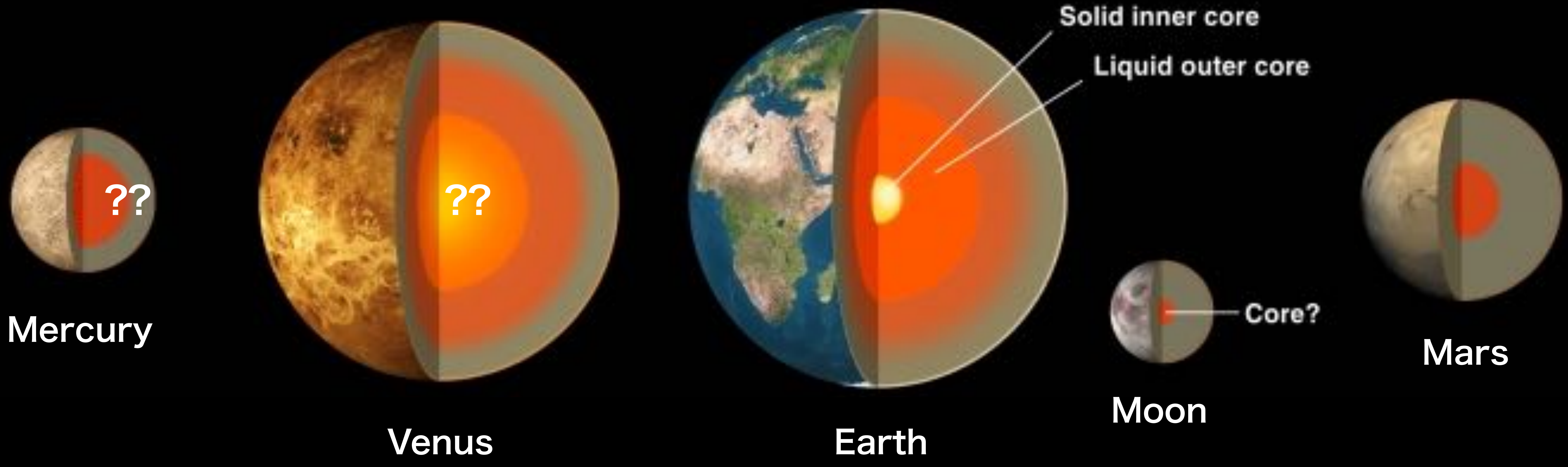


Image credit: NASA/LPI

How do we know planetary interiors?

Deep interior (> 10 km)

- **Bulk density**
- **Magnetic field (dynamo, crustal remnants, induced)**
- **Moment of inertia** (← precession, gravity field)
- **Gravity field (higher order)**
- **Seismology**
- Tidal deformation
- Analysis of materials originating from the deep interior (mantle xenolith)

Shallow interior (< 10 km)

- Radar observations, gamma ray and neutron spectrometry, muography, etc.

Seismology

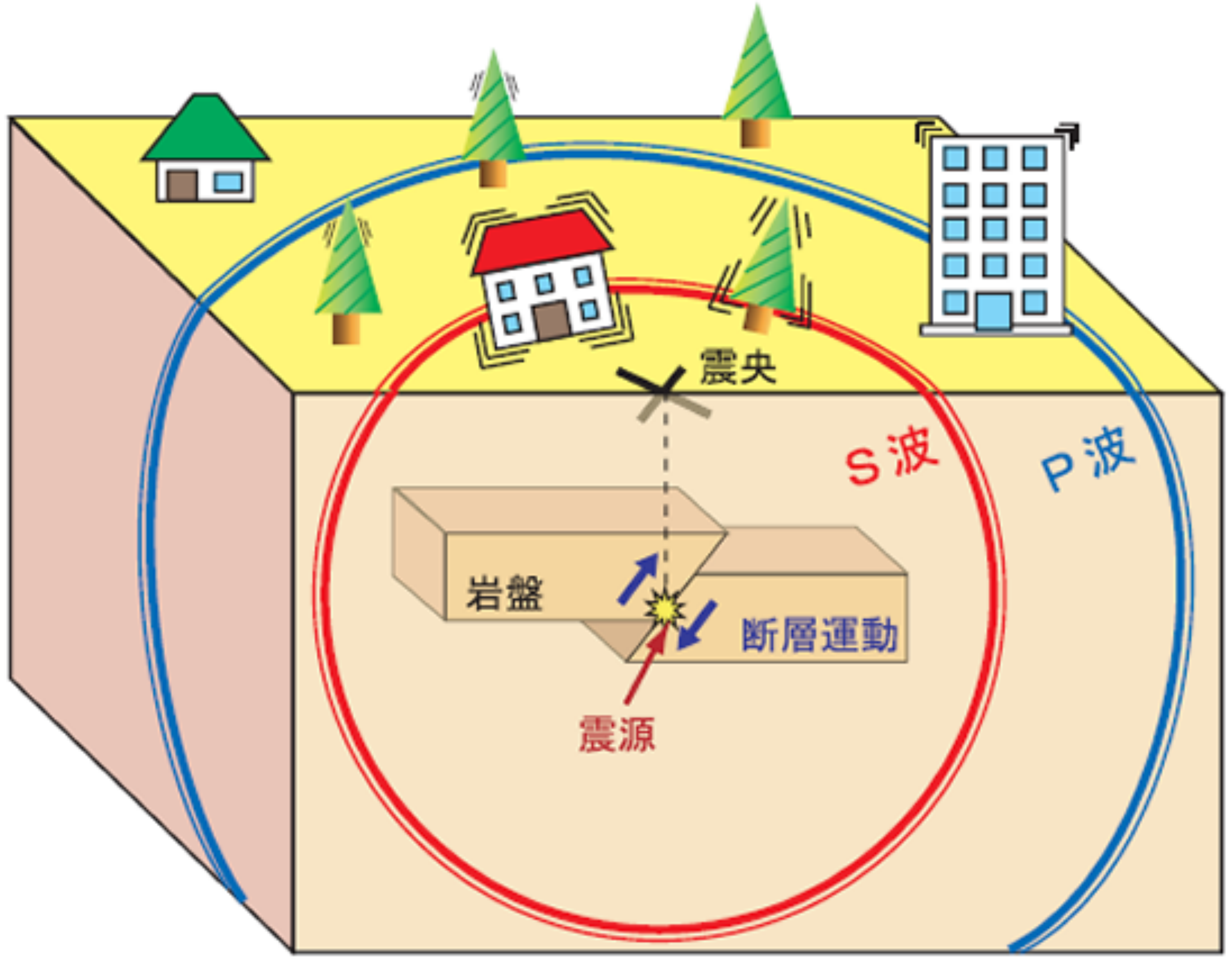


Image credit: 気象庁

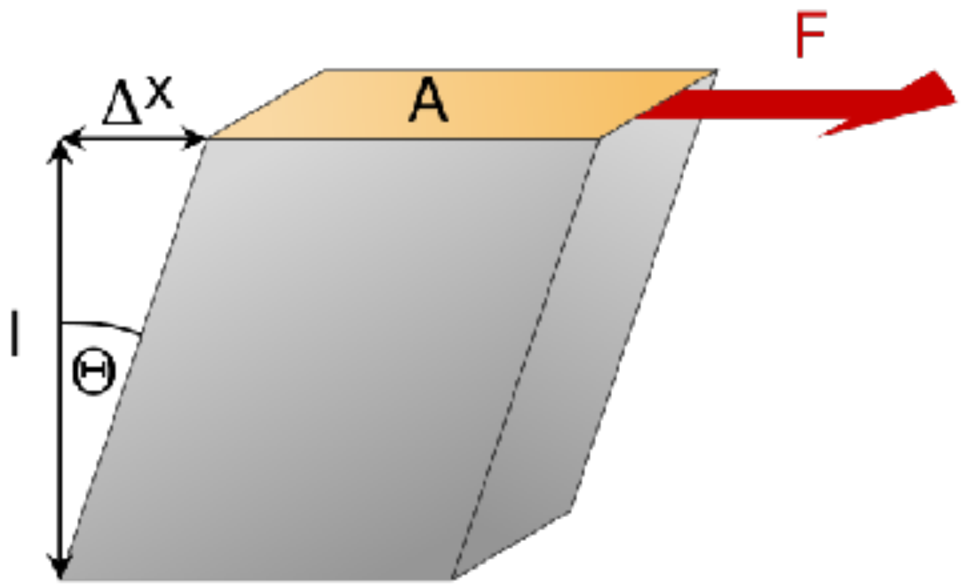
Speeds of primary and secondary waves ⇔ Material properties

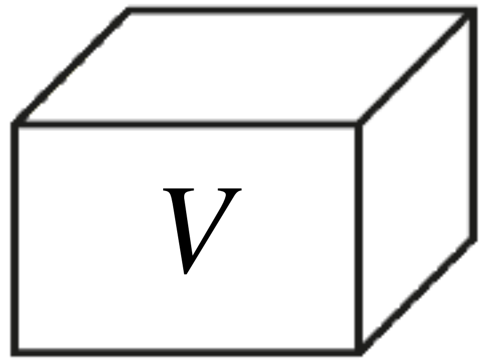
P-wave：$v_p = \sqrt{\dfrac{K + 4\mu/3}{\rho}}$ — (1)

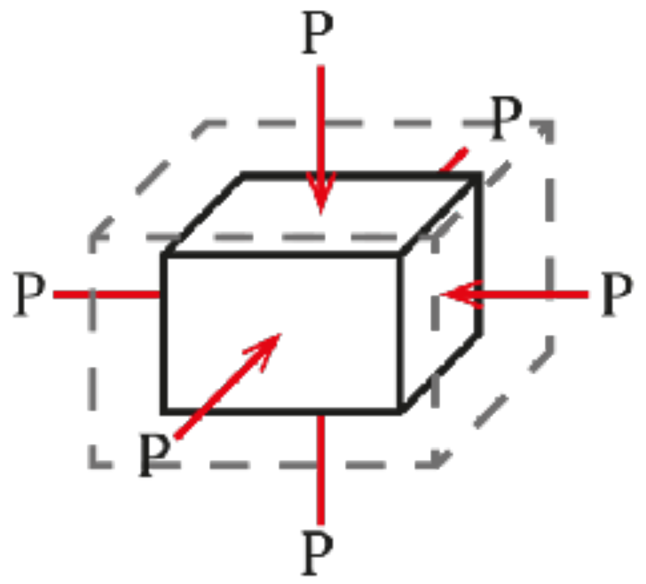
S-wave：$v_p = \sqrt{\dfrac{\mu}{\rho}}$ — (2)

where ρ: density,　μ: shear modulus, K: bulk modulus

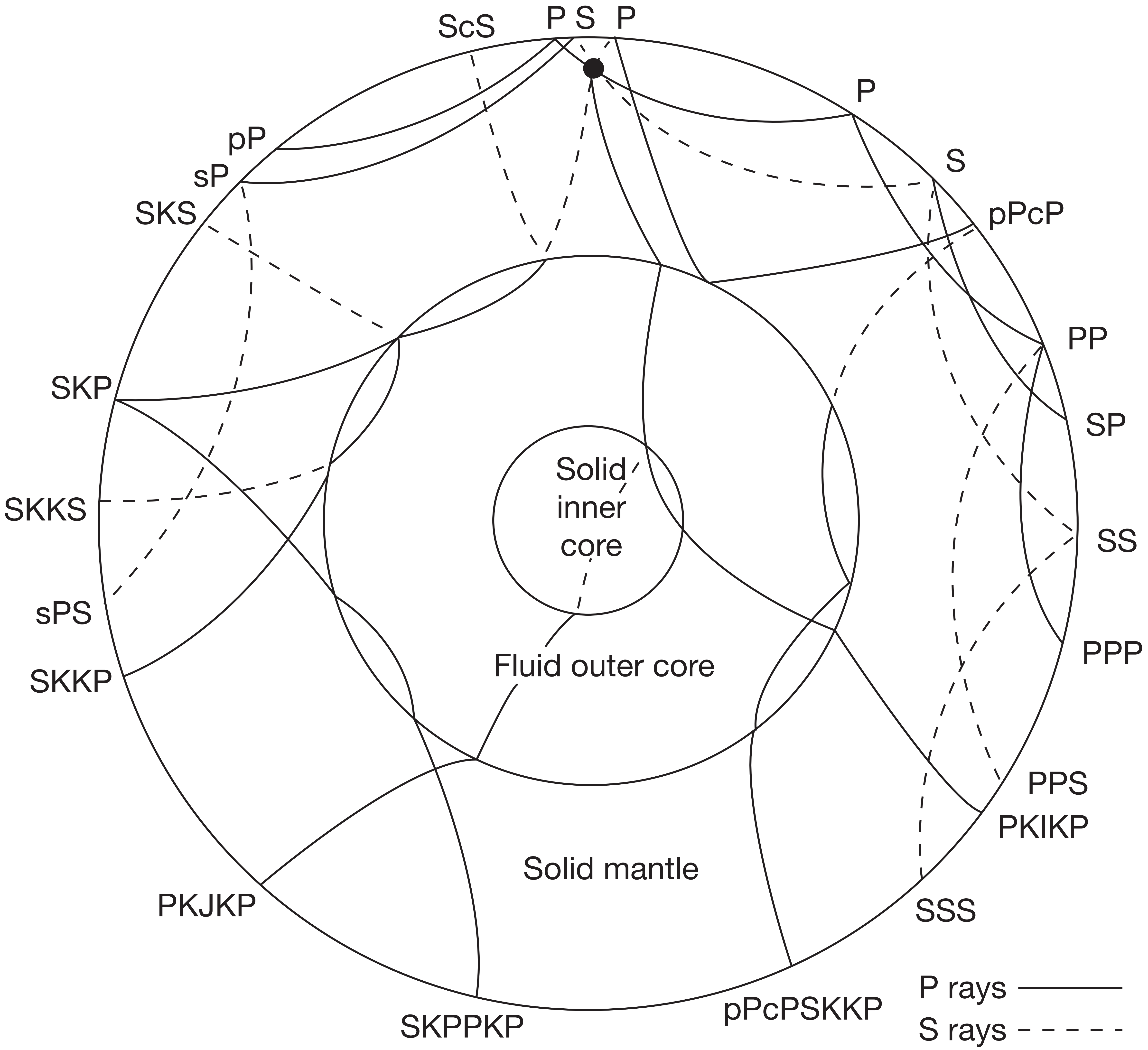
$\mu \equiv \dfrac{F/A}{\Delta x/l}$ — (3), $K \equiv -V\dfrac{\partial p}{\partial V}$ — (4)







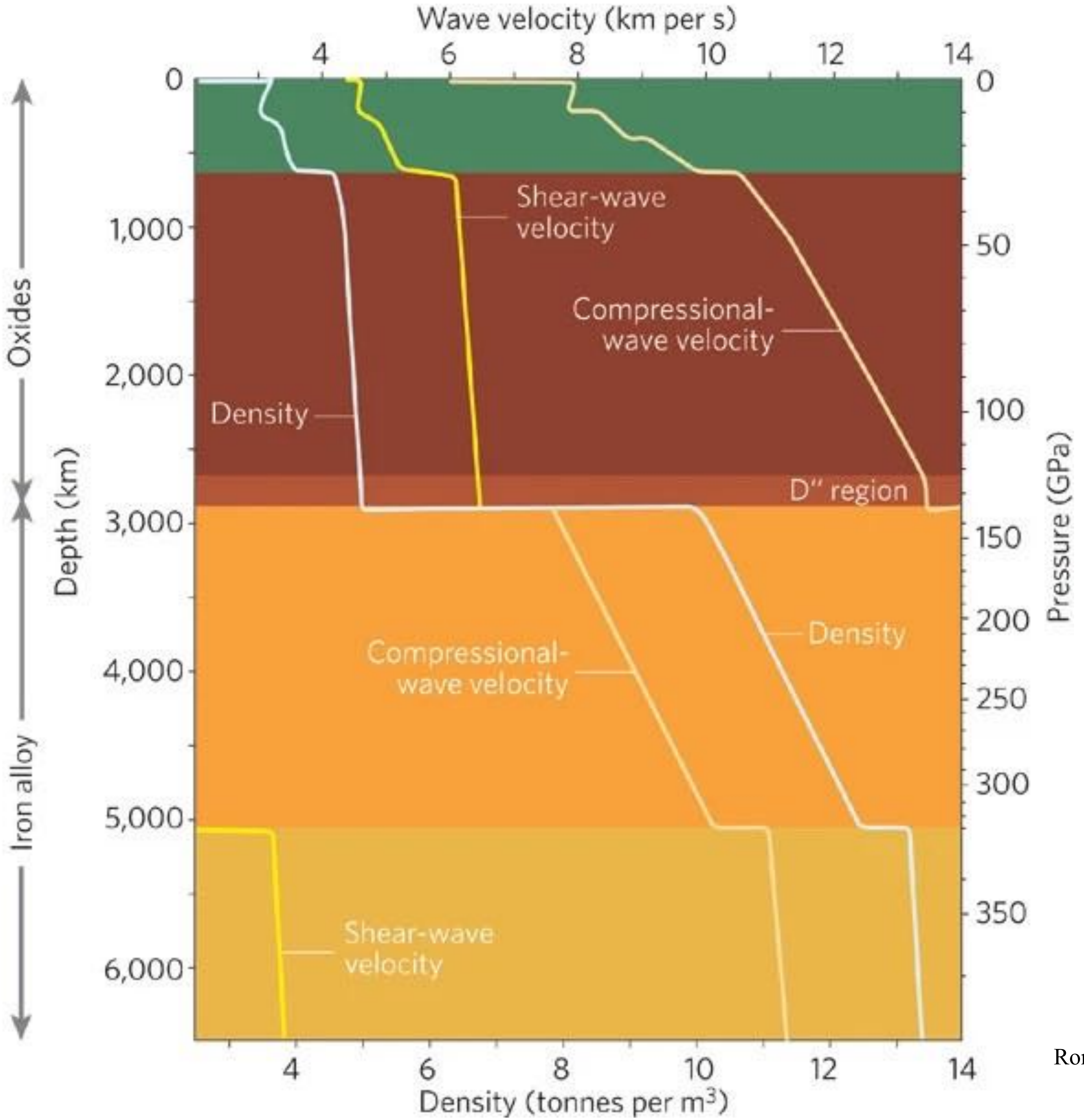
Propagation of seismic waves in the interior



P: P-wave (solid), S: S-wave (dashed)
K: P-wave in the outer core, I: P-wave in the inner core,
J: S-wave in the inner core,
c: reflection at the core-mantle boundary.

No S-wave in the outer core → liquid!

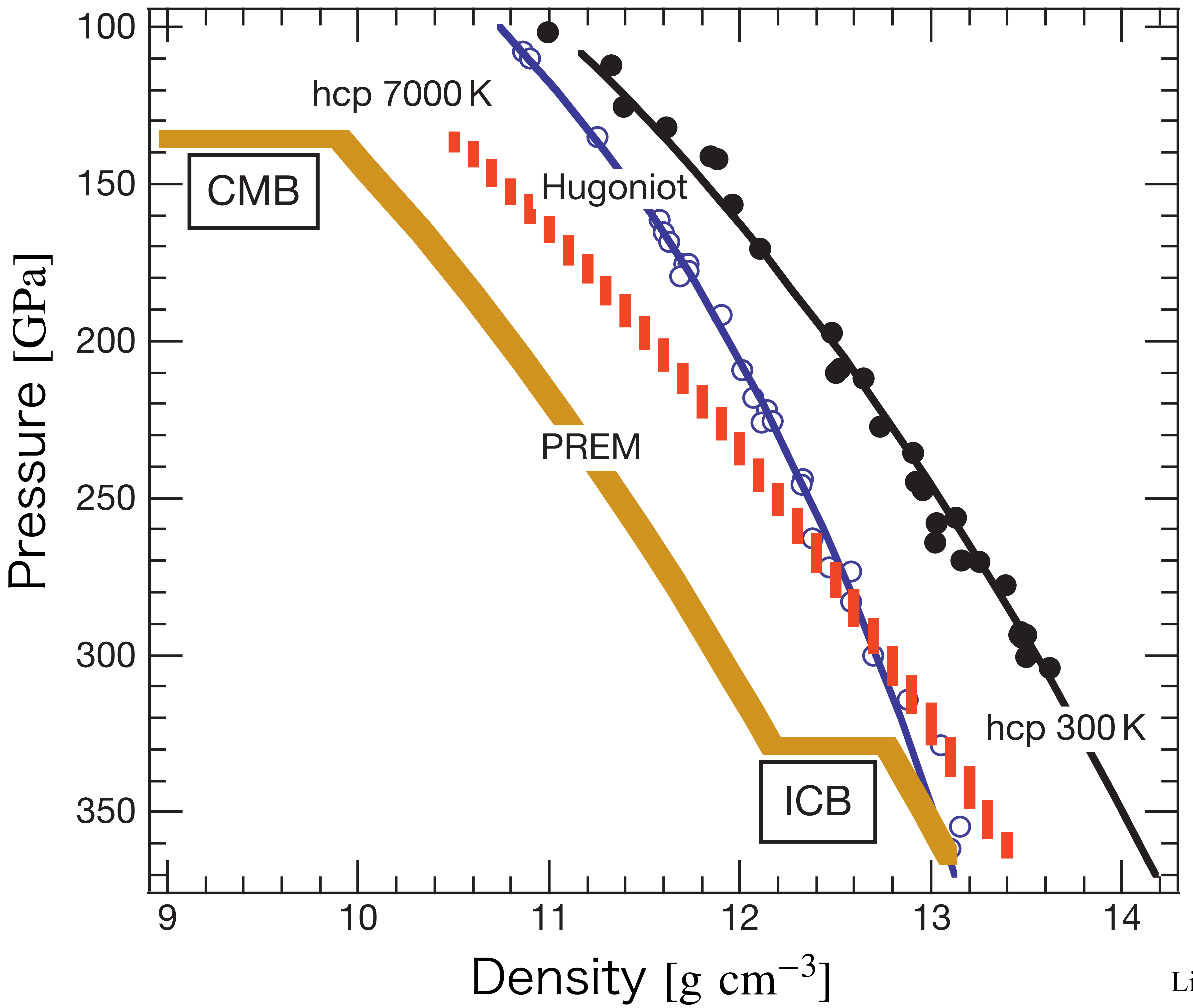
Dziewonski & Romanowicz (2015)
in *Treatise on Geophysics 2nd Edition*



Romanowicz (2008) *Nature*

Light elements in the core

Outer core density compared to that of pure Fe

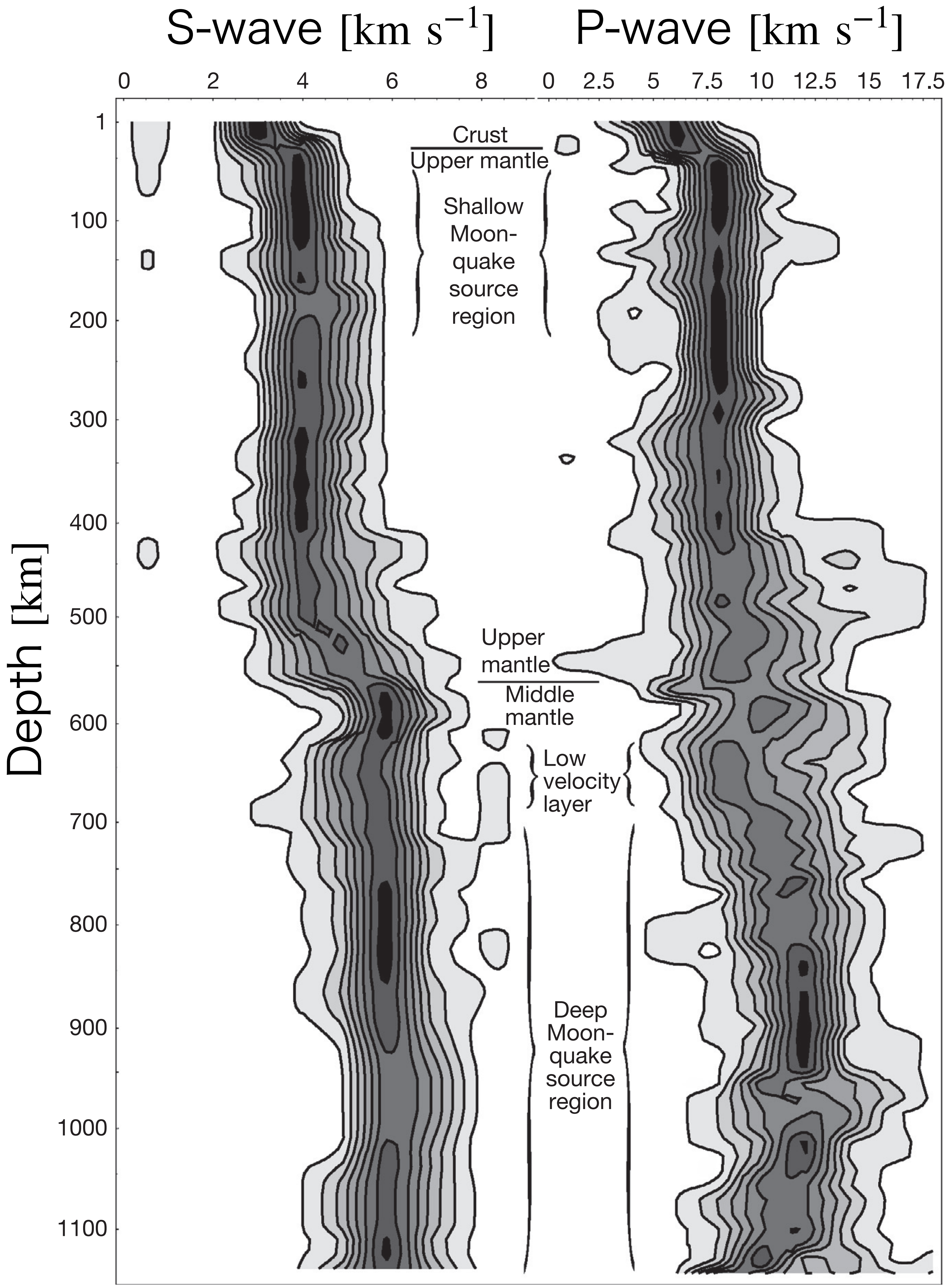


- The core is less dense than pure Fe by ~ 10 %
  → light element(s) – Si, S, O, H, and/or C?
- Total mass of the light elements in the core
  $\sim M_{\oplus} \times 0.33 \times 0.1 \sim 2 \times 10^{22}$ kg
  $\gg$ oceans (1.4×10^{21} kg), atmosphere (5.1×10^{18} kg)

Partitioning of light elements into the core has likely influenced determining Earth's surface environment

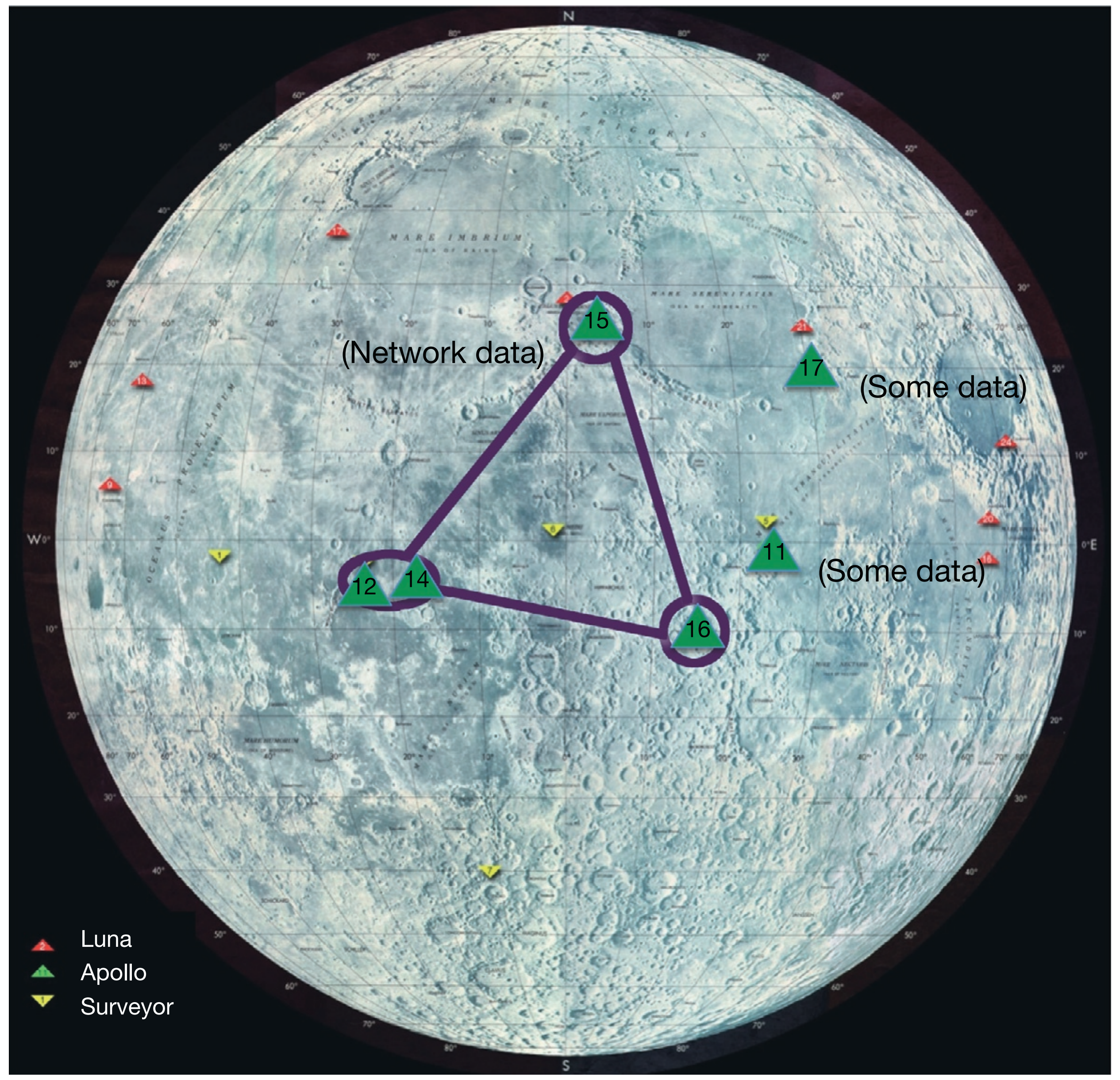
Li & Fei (2014) *Treatise on Geochemistry 2nd Edition*

Lunar seismology



Khan & Mosegaard (2002) *J. Geophys. Res.*

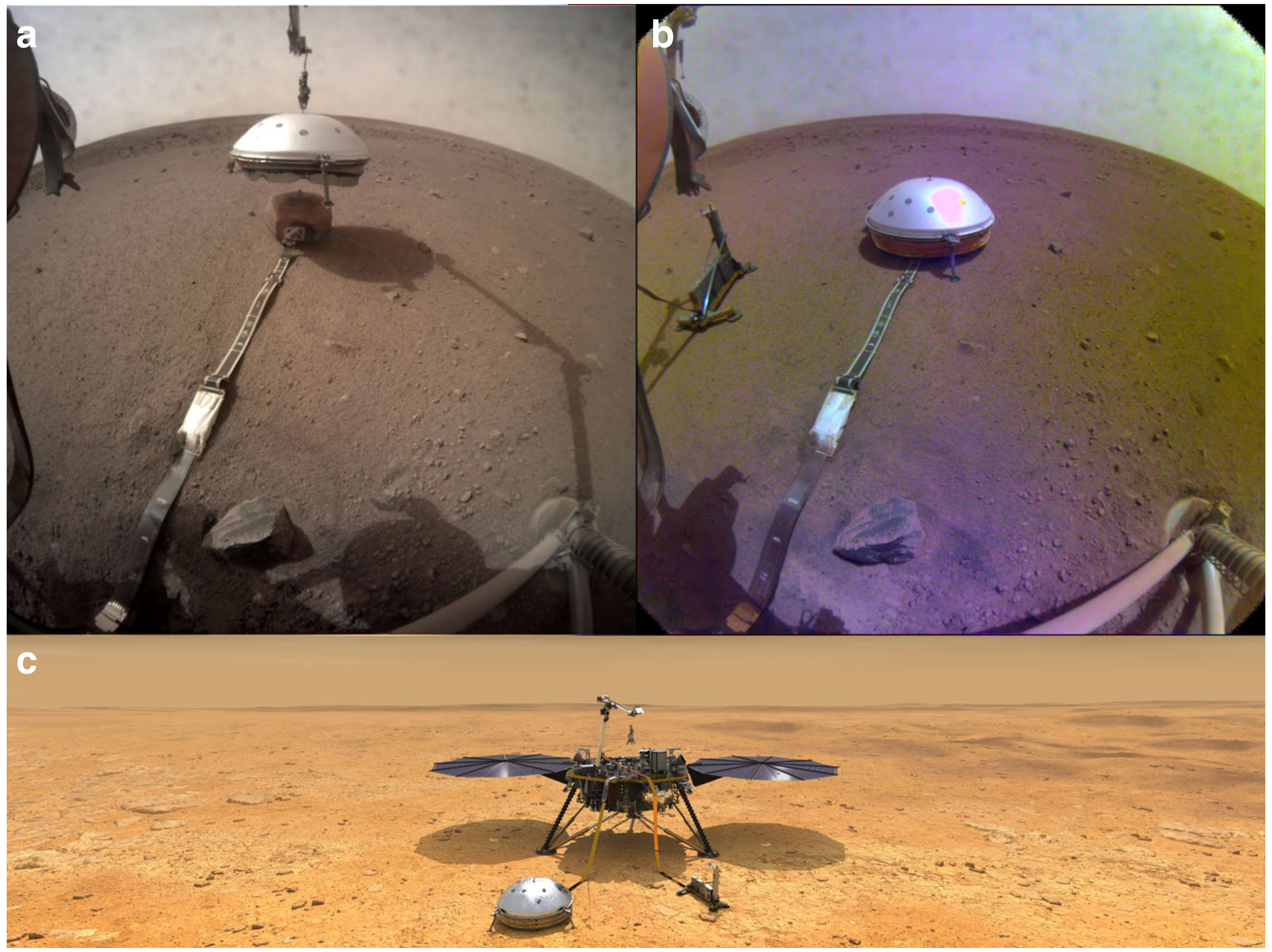
- Seismograph network at Apollo 12, 14-16 landing sites
- Though data is limited compared to those on Earth, Lunar interior structure has been constrained by the seismology
  - Core size ~170–360 km (Nakamura et al. 1974)
    - Consistent with the estimates from the moment of inertia and induced magnetic fields

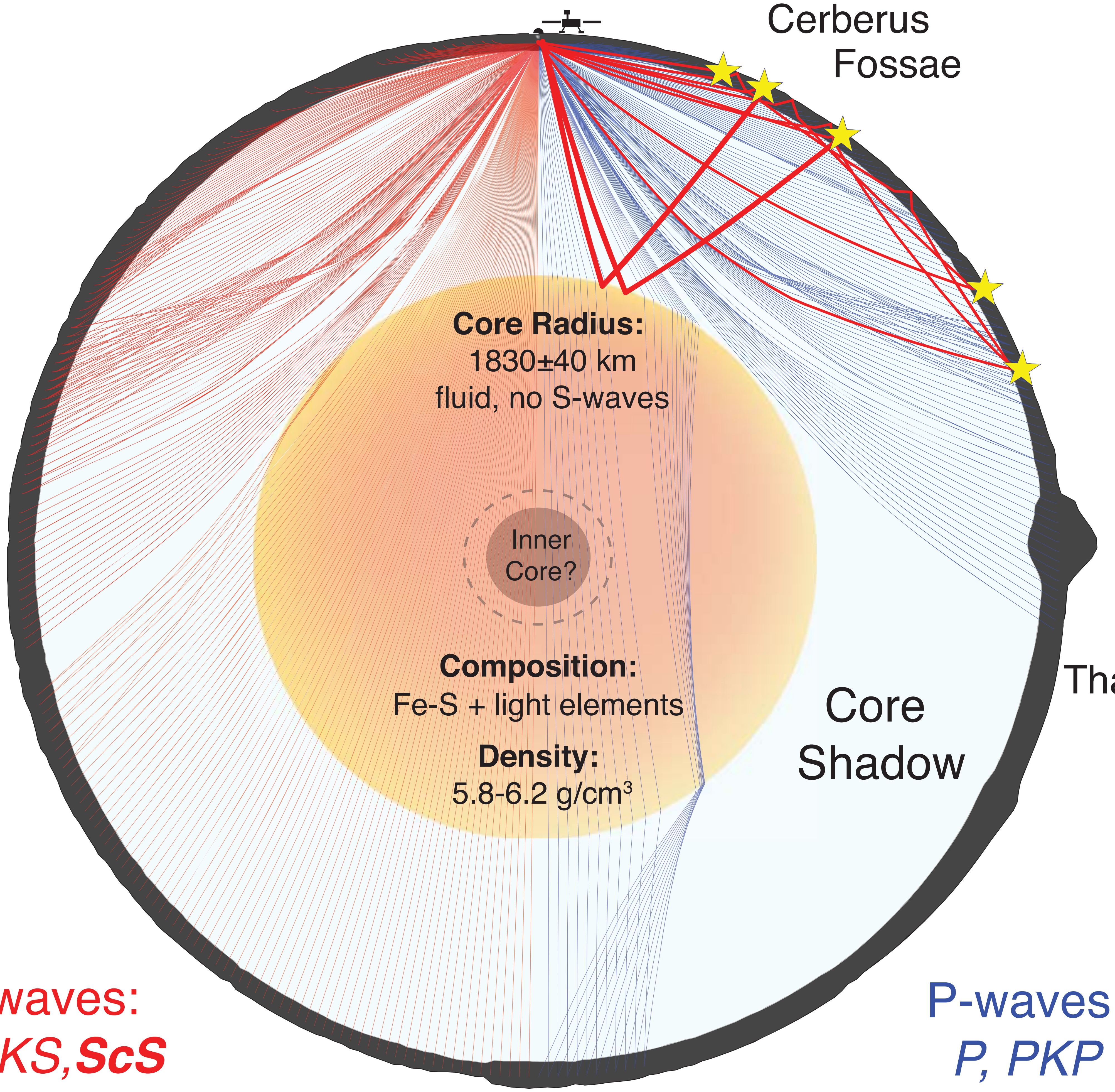


Lunar seismograph network
Lognonné & Johnson (2015)
in *Treatise on Geophysics 2nd Edition*

Mars' seismology

NASA's InSight and its seismograph (Image credit: NASA/JPL-Caltech)





First detection of reflection at the CMB

→ Core is large ($R_c/R = 0.54 \pm 0.01$),

and has low density ($\rho = 5{,}800\text{-}6{,}200\ \mathrm{kg/m^3}$)

Stähler et al. (2021) *Science*

Interiors of gas giants

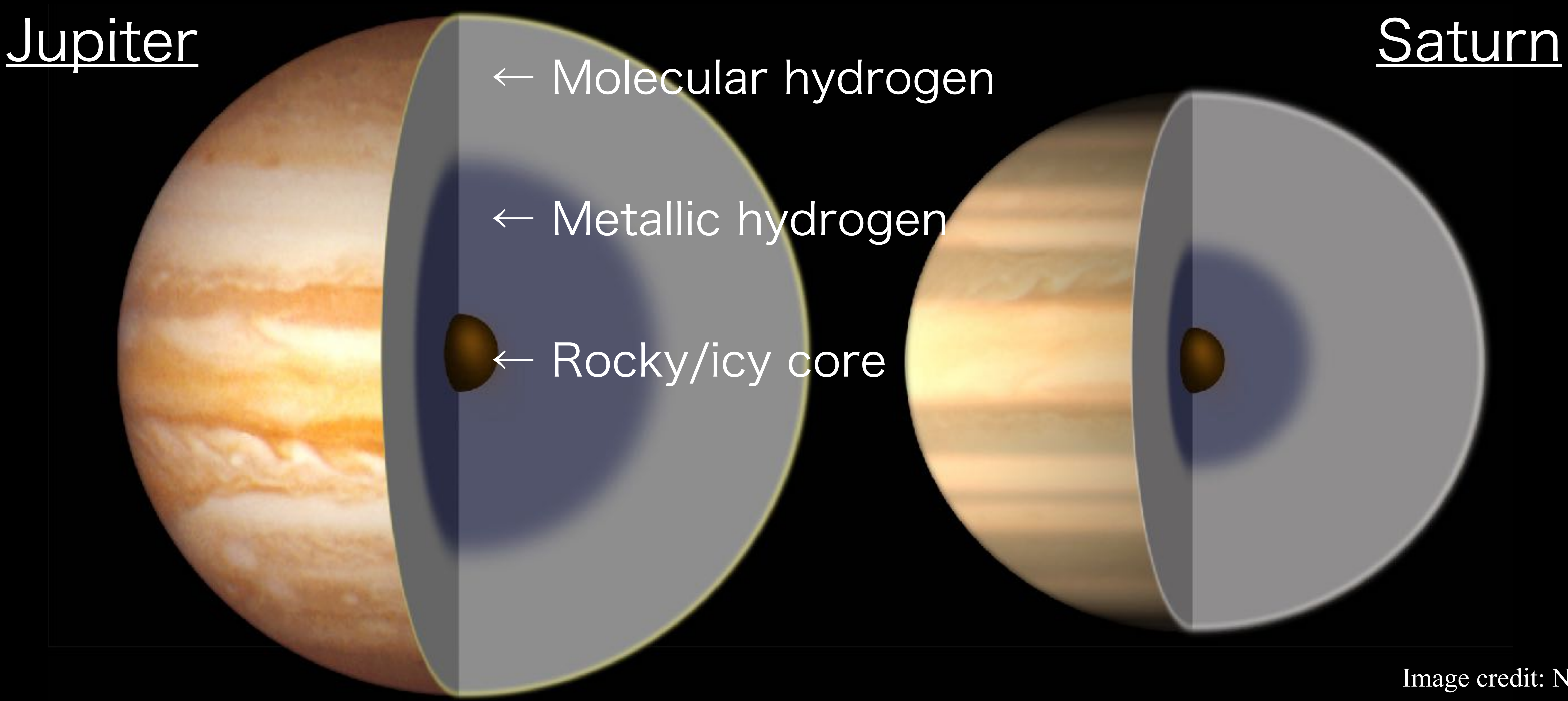
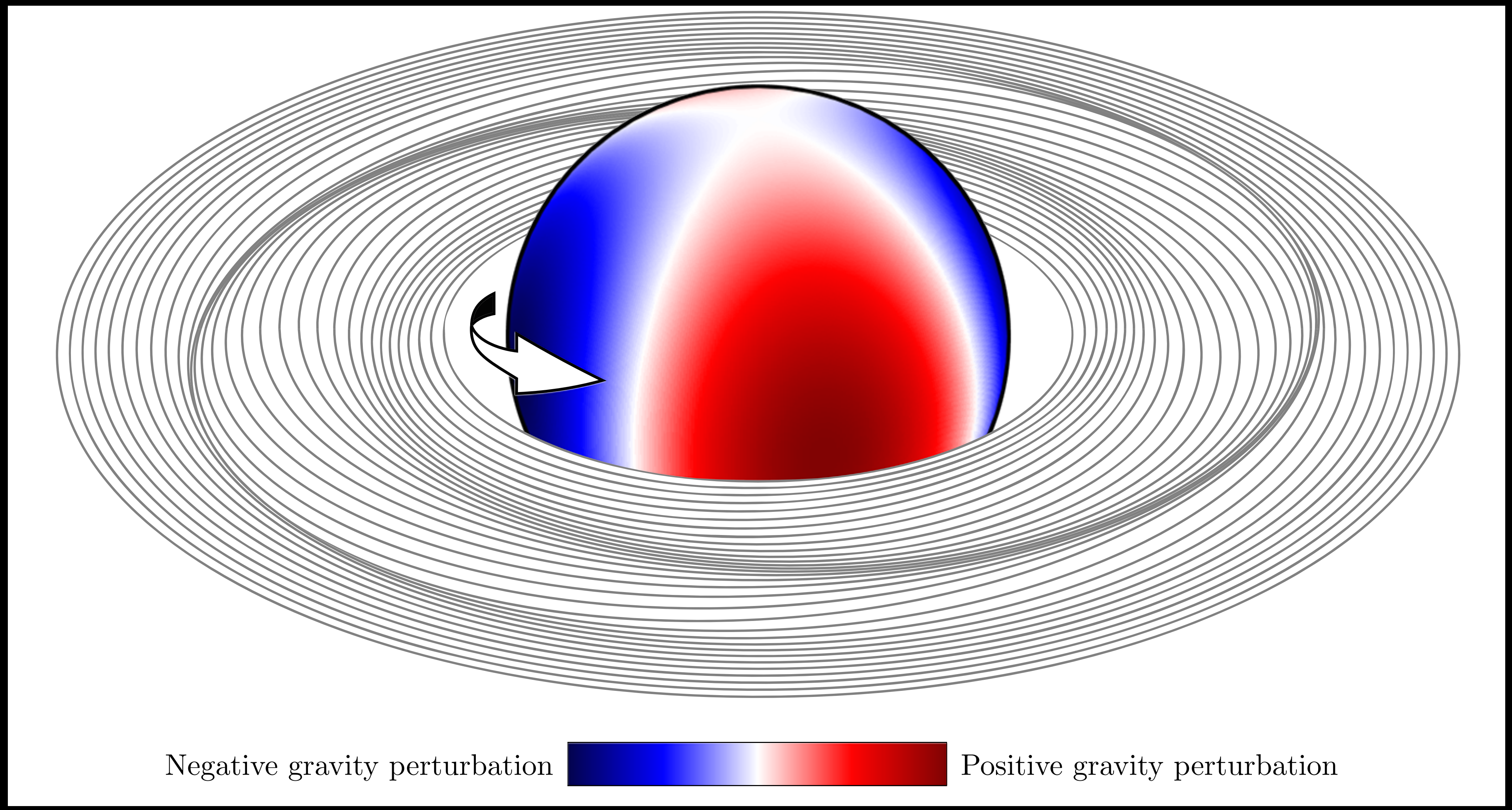


Image credit: NASA/LPI

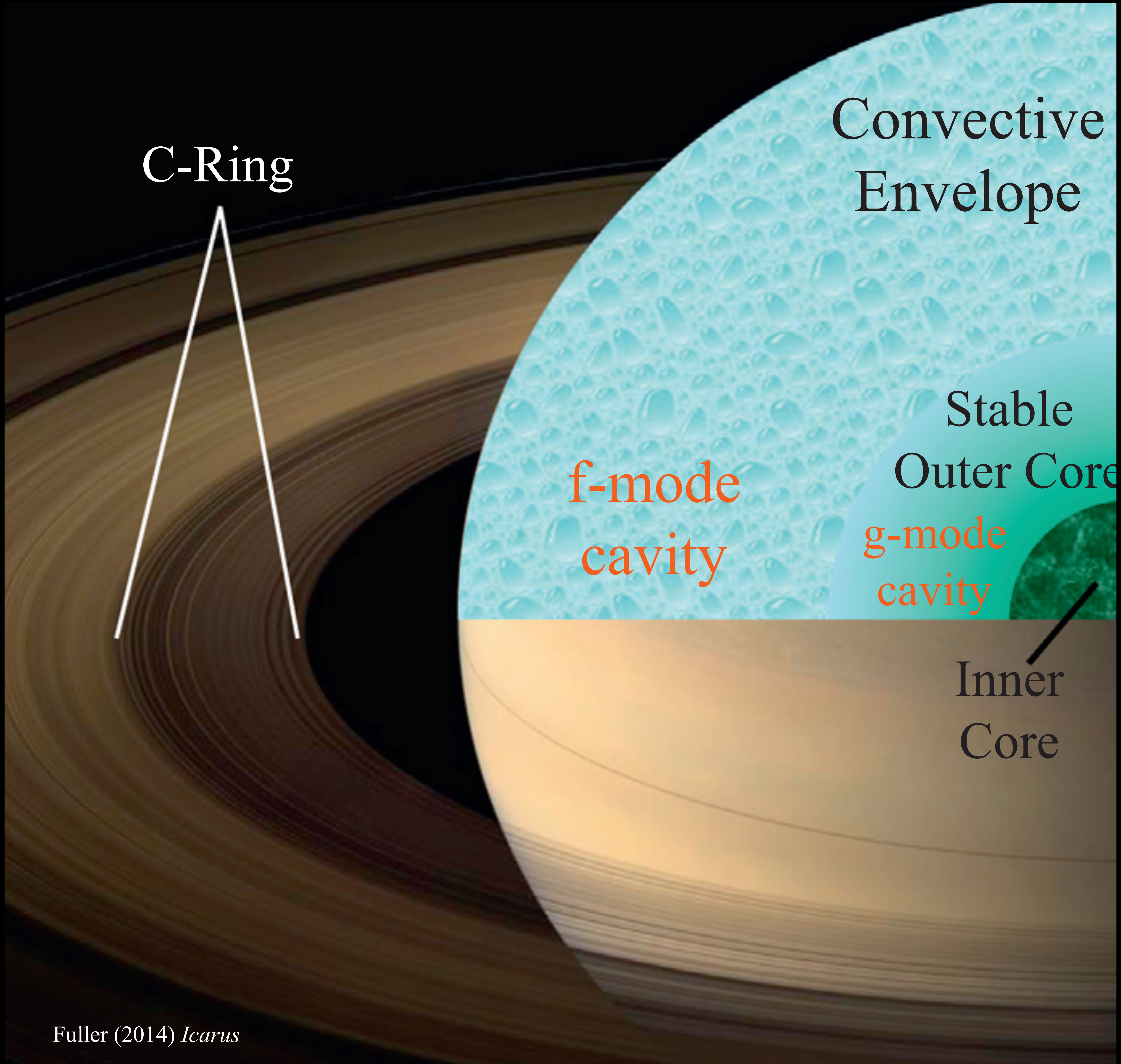
- The majority of the mass ($\simeq 70 - 90\%$) is hydrogen (H) and helium
- High pressure (>100 GPa) interiors → Metallic H (with free electrons) → Magnetic dynamo

Seismology of gas giants

Saturn's interior informed by waves on the ring



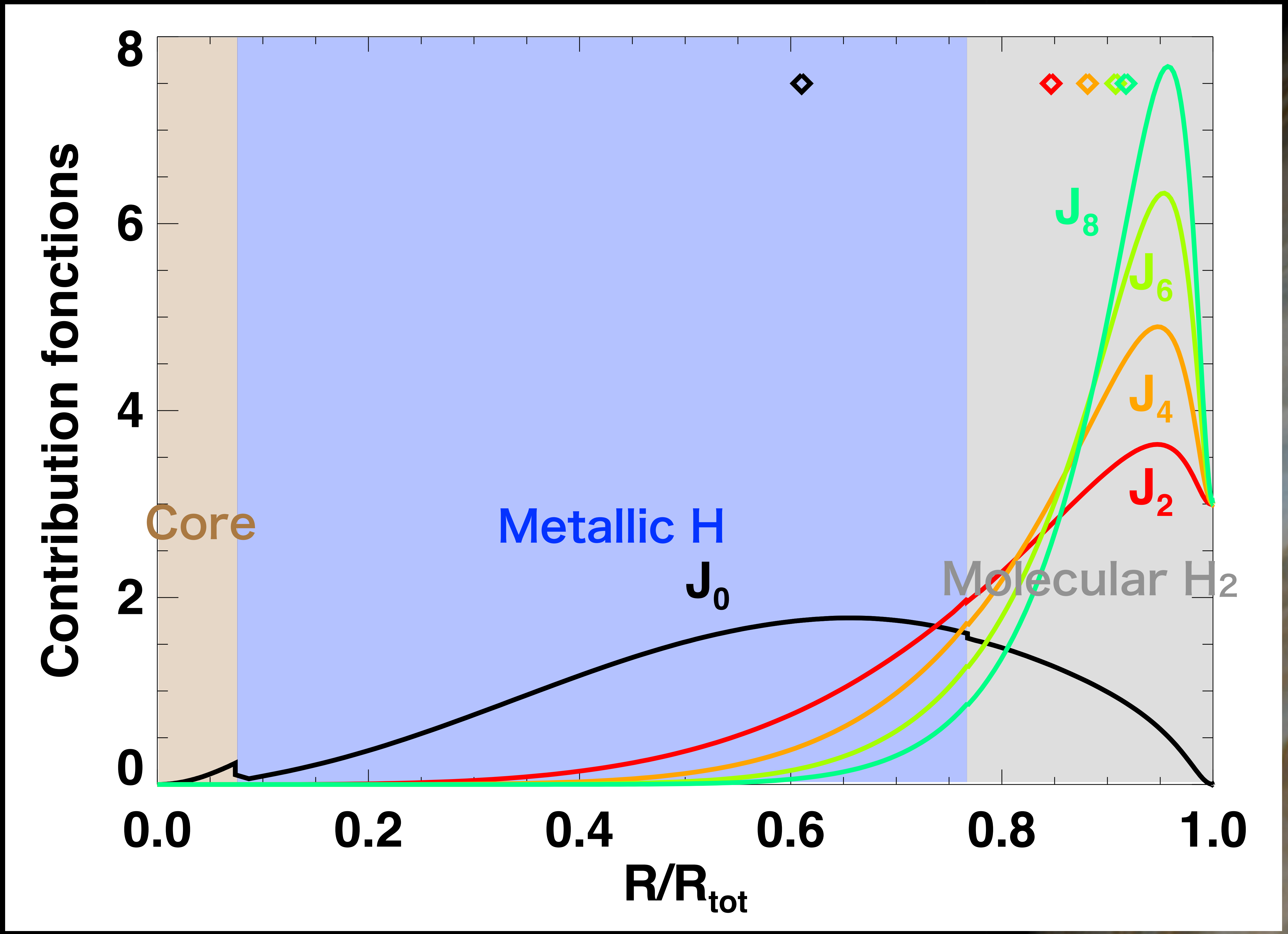
Mankovich (2020)



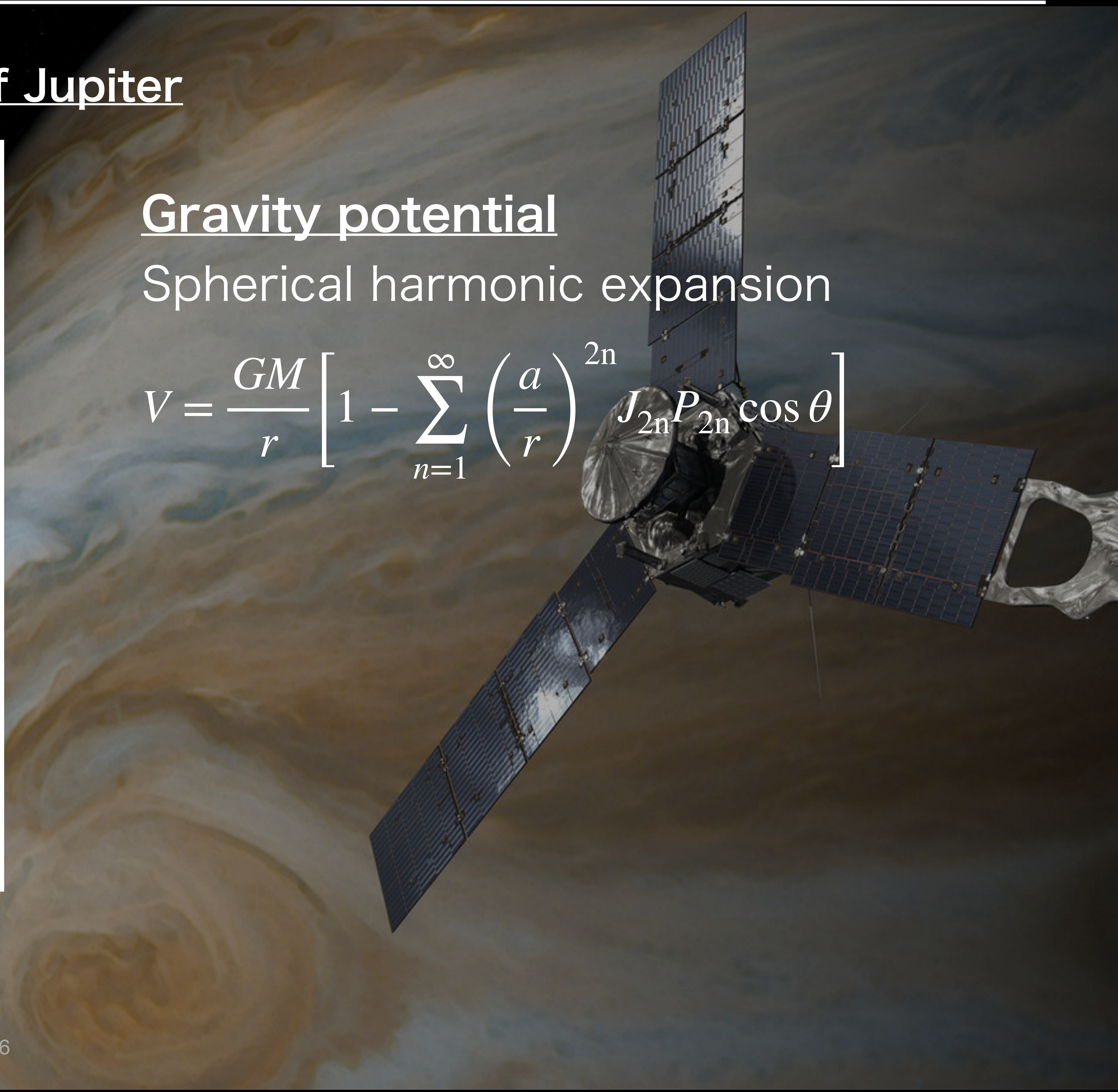
Fuller (2014) *Icarus*

Gravity measurements

Contributions to the gravitational moments of Jupiter



Guillot & Gauter (2014)



Gravity potential

Spherical harmonic expansion

$$V = \frac{GM}{r}\left[1 - \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n} J_{2n}P_{2n}\cos\theta\right]$$

Interiors of icy giants

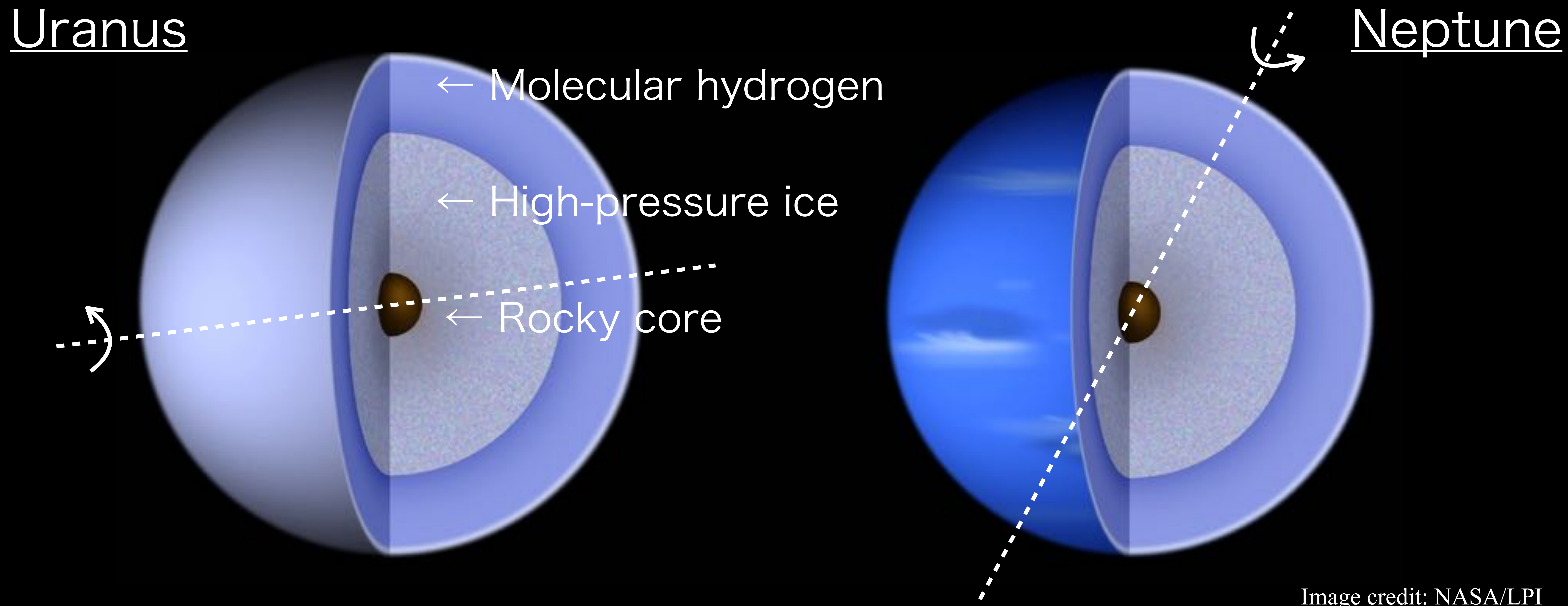


Image credit: NASA/LPI

- High-pressure ice: Super-ionic phase (protons behave like free electrons)
- Interiors less understood (no orbiter measurements so far)

Moons

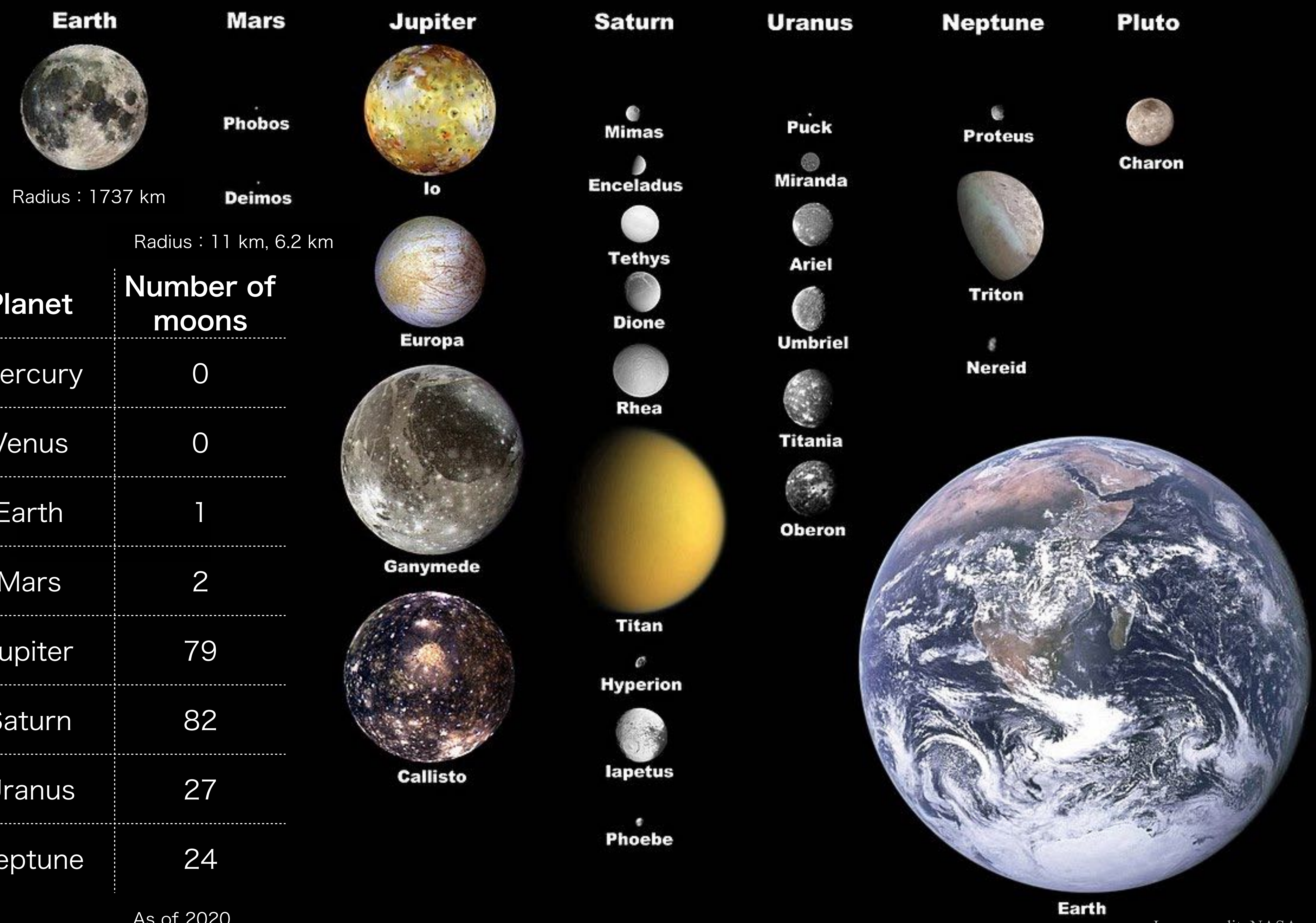
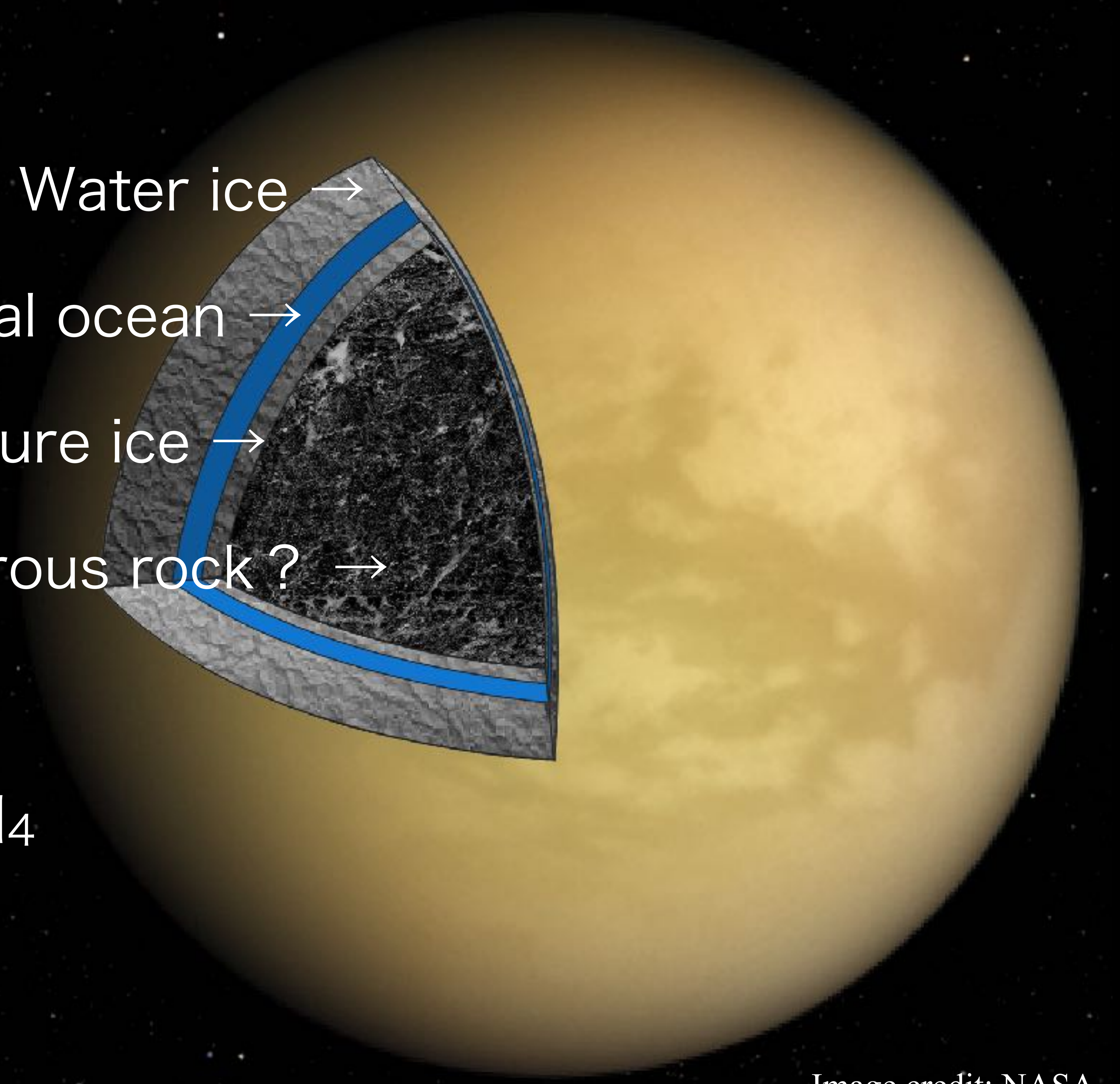


Image credit: NASA

Planet	Number of moons
Mercury	0
Venus	0
Earth	1
Mars	2
Jupiter	79
Saturn	82
Uranus	27
Neptune	24

As of 2020

Saturn's icy moon: Titan



- Atmosphere: ~1.5×10^5 Pa, N_2 + a few % CH_4
- Photochemical haze (organic molecules)
- Lakes: CH_4, C_2H_6 liquid
- Internal ocean: H_2O (common in many icy moons)

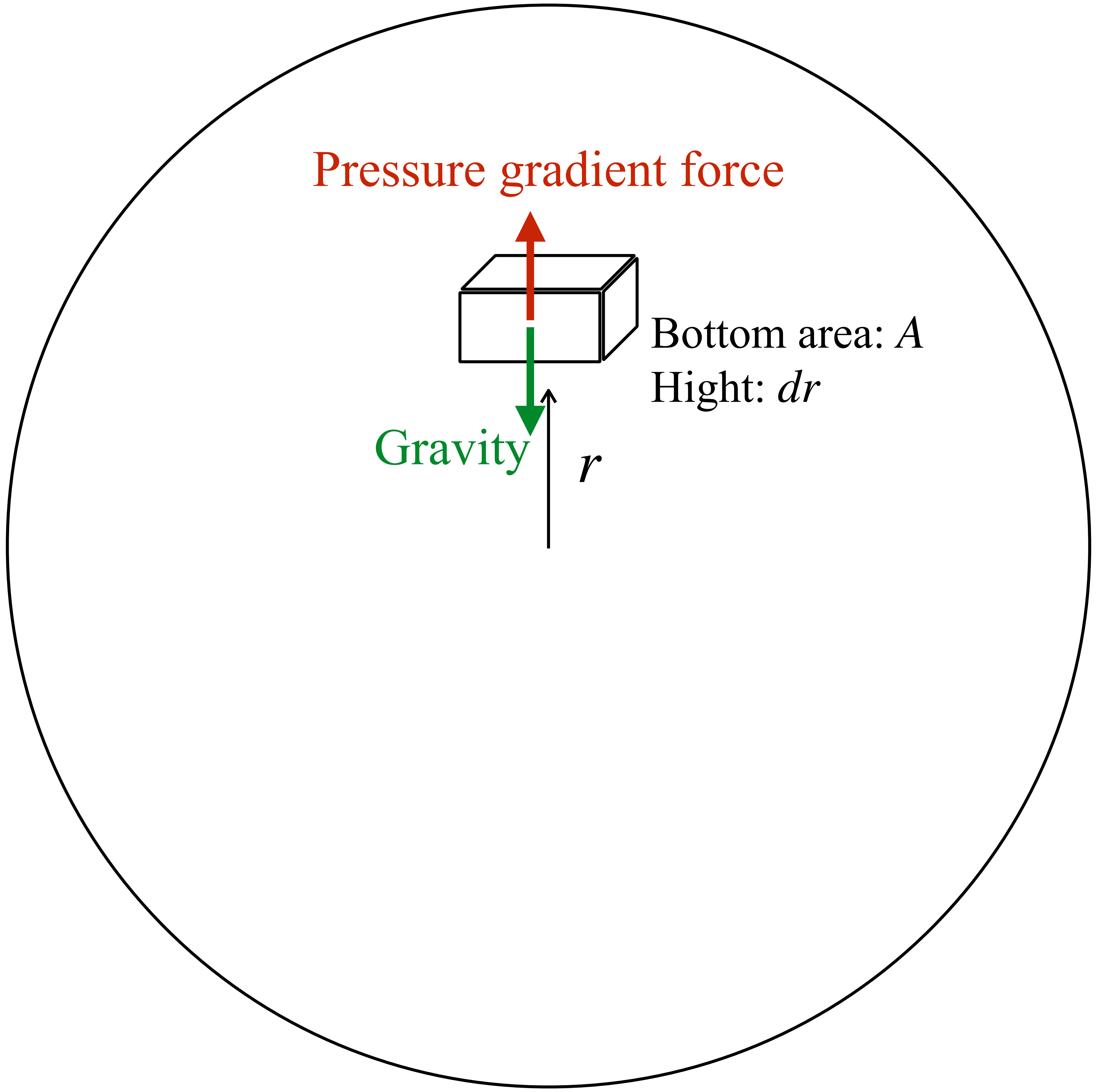
Image credit: NASA

DragonFly (launch scheduled in 2027)

(Image credit: NASA)

- Astrobiological measurements
- **Seismograph!**

Equation for hydrostatic equilibrium



We consider the equilibrium of the forces.

Pressure gradient force: $p(r)A - p(r+dr)A = -\frac{dp}{dr}dr \cdot A$ — (1)

Gravity: $-\rho A dr \cdot g(r) = -\rho A dr \cdot \frac{GM_r}{r^2}$ — (2)

From (1) + (2) = 0, we obtain,

$$\frac{dp}{dr} = -\rho \frac{GM_r}{r^2}$$ — (3) : The hydrostatic equilibrium equation

p: pressure, ρ: density, g: gravity,

$M_r(r)$: enclosed mass within the sphere of the radius r

An example: pressure change in the ocean

Let us think about pressure change when diving in the ocean.

The hydrostatic equation is given as,

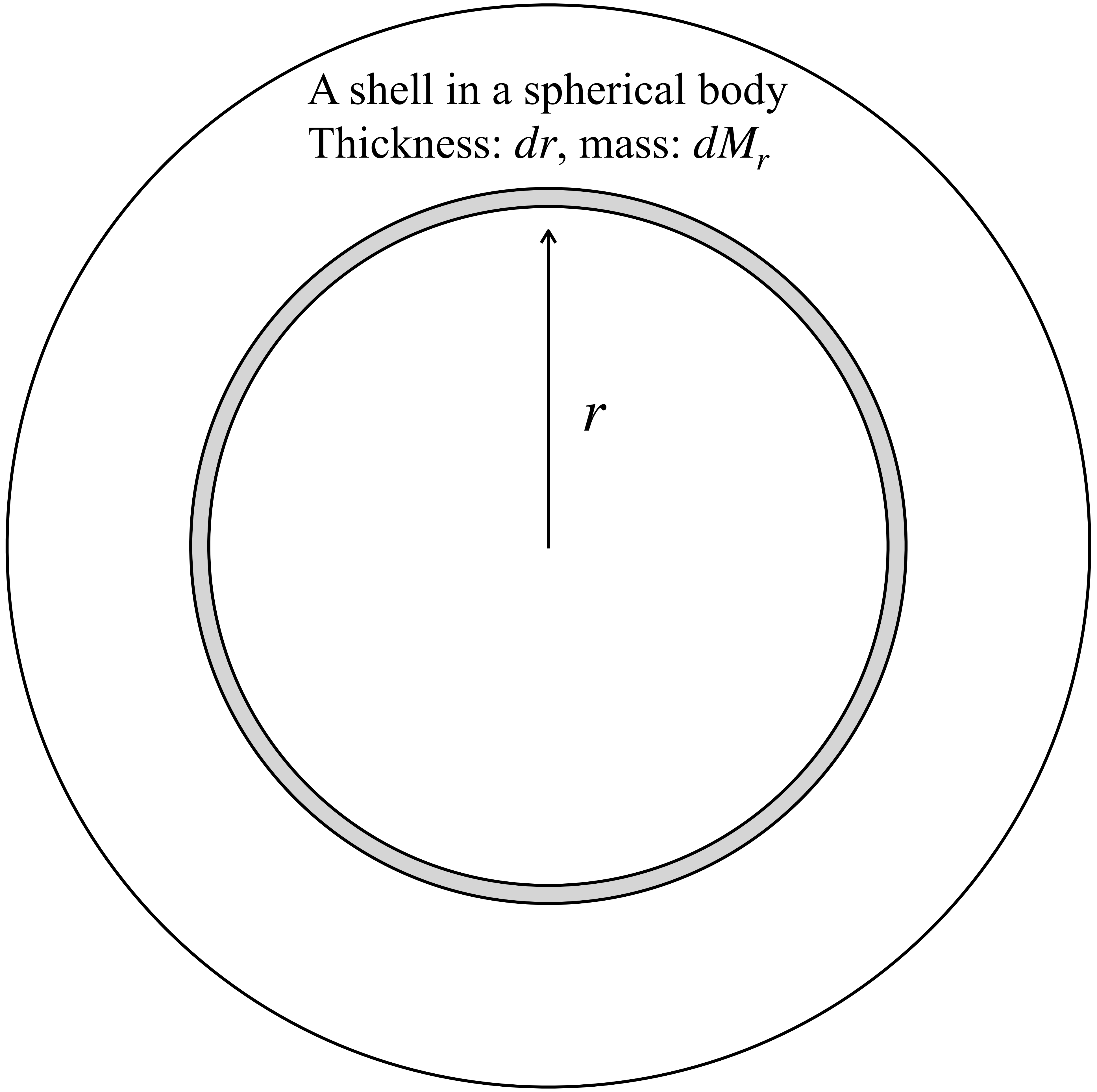
$$\frac{dp}{dr} = -\rho\frac{GM_r}{r^2} \simeq \rho g \quad \text{— (4)},$$

where g is the gravitational acceleration at the surface ($9.8\ \mathrm{m\ s^{-2}}$).

Given the density of water at $1\ \mathrm{atm}$, $\rho = 10^3\ \mathrm{kg\ m^{-3}}$, we obtain,

$$\frac{dp}{dr} = -10^3\ \mathrm{kg\ m^{-3}} \cdot 9.8\ \mathrm{m\ s^{-2}} \simeq -10^4\ \mathrm{Pa\ m^{-1}} \simeq -1\ \mathrm{atm}/10\ \mathrm{m} \quad \text{— (5)}$$

Equation for mass conservation



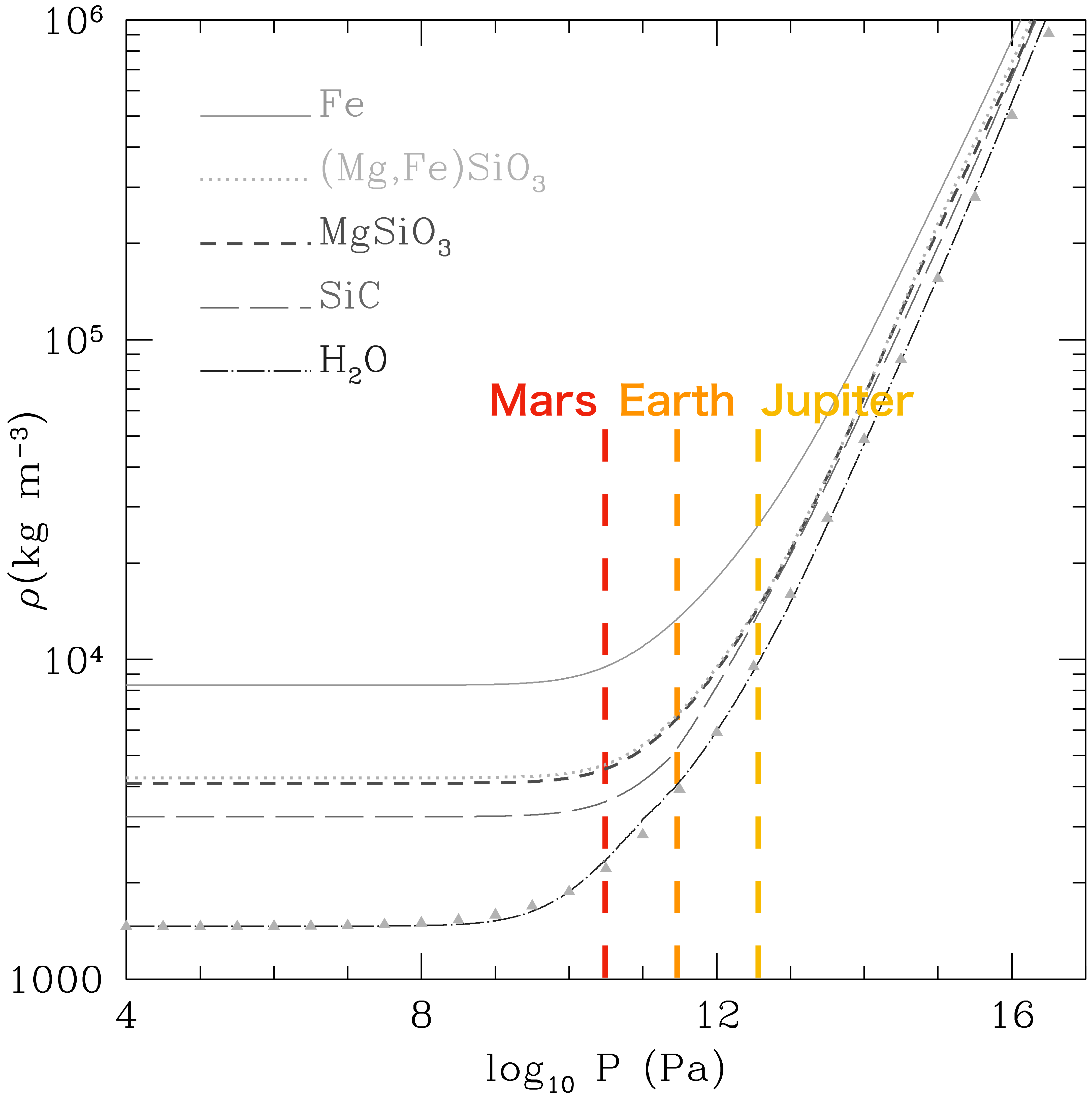
The mass of the shell, dMr, is given by,

$dM_r = 4\pi r^2 dr \cdot \rho$ — (1)

$\therefore \dfrac{dM_r}{dr} = 4\pi r^2 \rho$ — (2) : The mass conservation equation

Equation of state

$P-\rho$ relations for solid materials given by the equation of state



Seager et al. (2007) *Astrophys. J.*

An equation of state (EoS) is a function which relates thermodynamic variables: p, ρ, T

Examples

- Ideal gas low: $p = \dfrac{\rho k_B T}{m}$ — (1)
- (Third-order) Birch-Murnagham EoS:

$$p = \frac{3}{2}K_0(\eta^{7/3} - \eta^{5/3})\left[1 + \frac{3}{4}(K_0' - 4)(\eta^{2/3} - 1)\right] \quad \text{— (2)}$$

Here $\eta = \rho/\rho_0, K$ is the bulk modulus, K' is the pressure derivative, and the subscript 0 stands for the value at the ambient conditions

Structure equations for a spherically-symmetric body

Equations solved for four variables – $M_r(r)$, $p(r)$, $\rho(r)$, $T(r)$

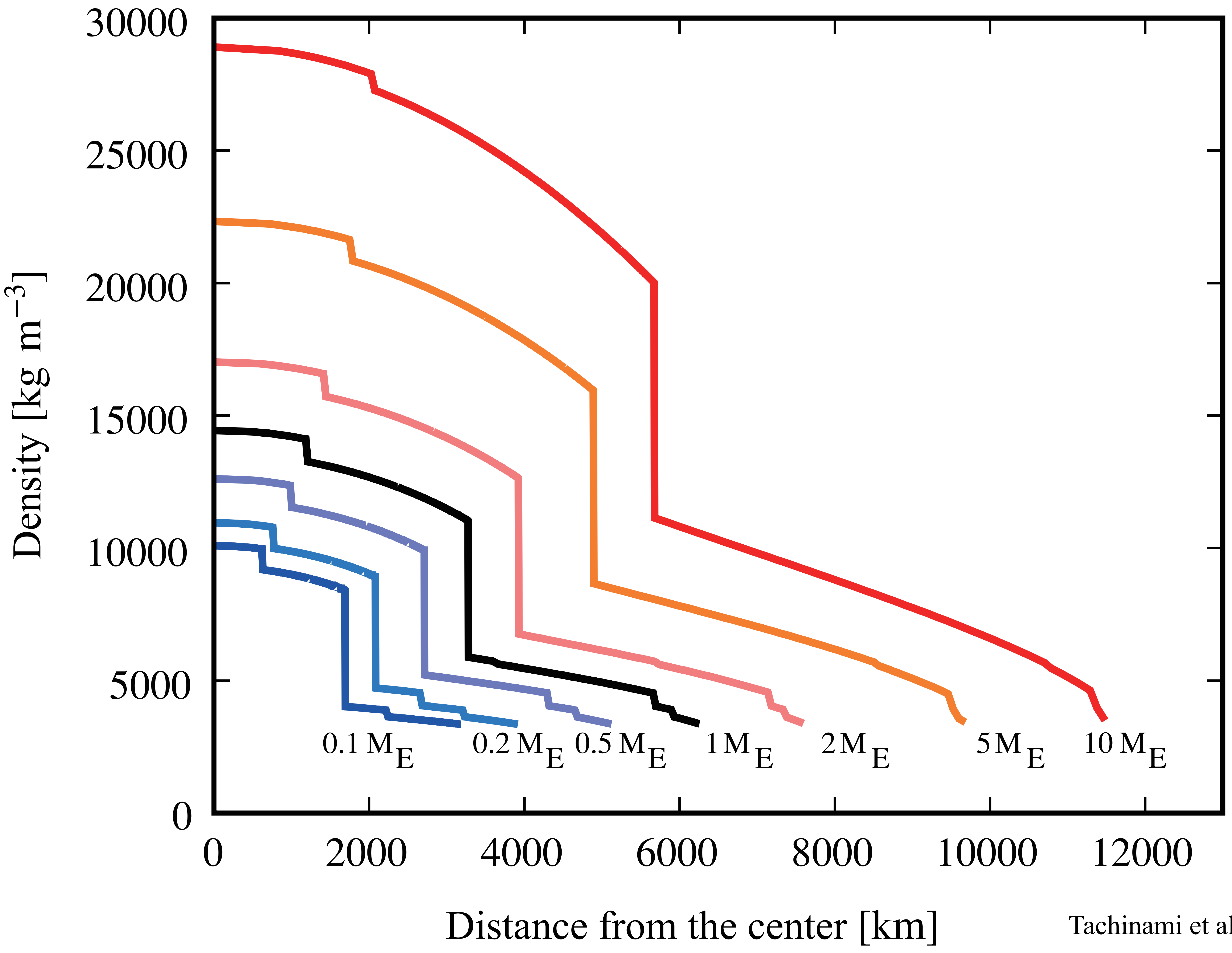
- Hydrostatic equilibrium: $\frac{dp}{dr} = -\rho \frac{GM_r}{r^2}$ — (1)
- Mass conservation: $\frac{dM_r}{dr} = 4\pi r^2 \rho$ — (2)
- Equation of state: $p = f(\rho, T)$ — (3)
- Energy transfer: $\frac{dT}{dr} = -\min\left(\left|\left(\frac{dT}{dr}\right)\right|_{\rm cond}, \left|\left(\frac{dT}{dr}\right)\right|_{\rm rad}, \left|\left(\frac{dT}{dr}\right)\right|_{\rm conv} \right)$ — (4)

Because the temperature effect on Eq. 3 is minor for solid bodies ($p \simeq f(\rho)$), Eqs. are closed with 1–3.

$\because$ Thermal expansion coefficient for mantle material $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \sim 10^{-5}\ {\rm K}^{-1}$

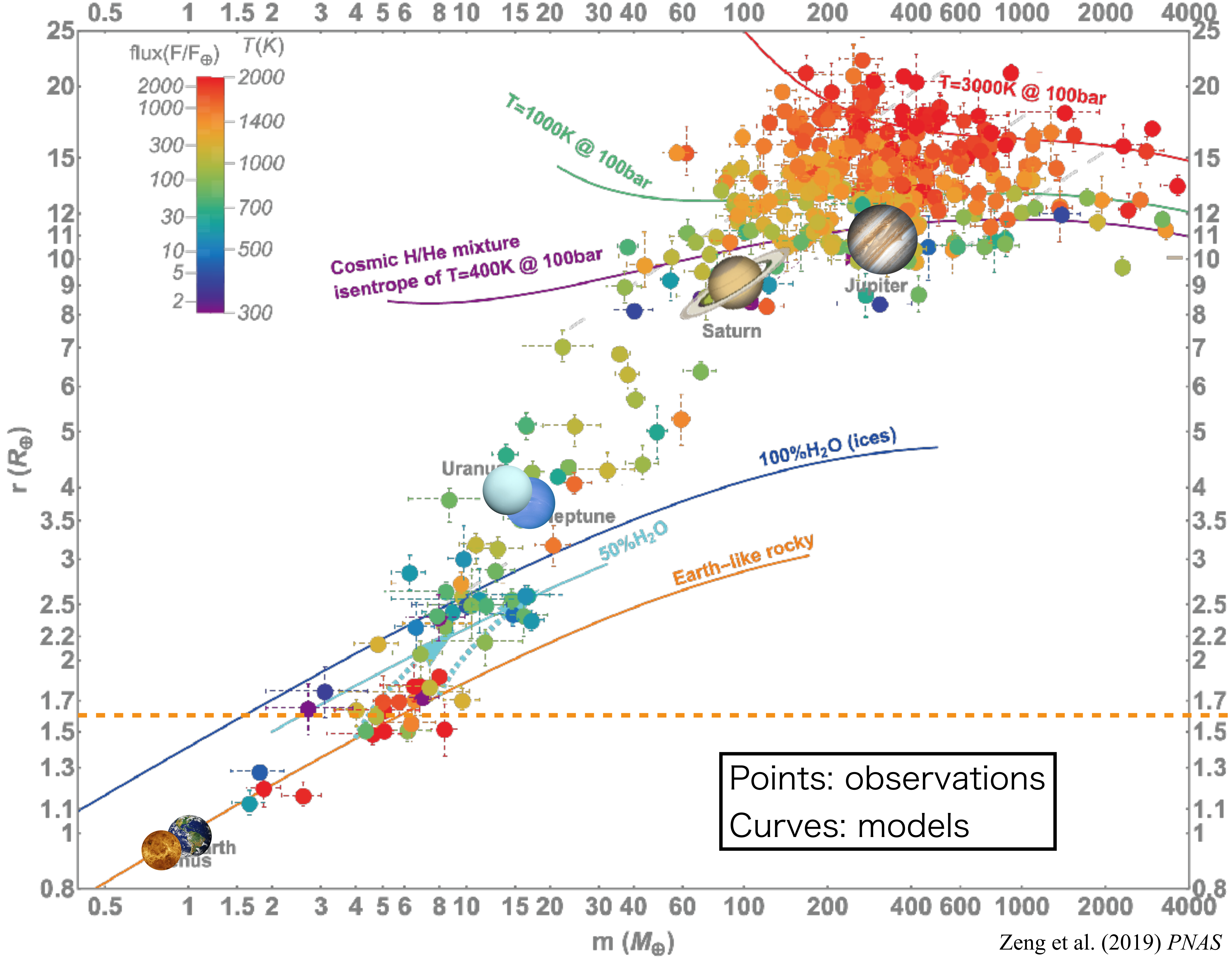
$\rightarrow$ volume change is only $\sim 1\,\%/10^3$ K

The model results for Earth-like planets

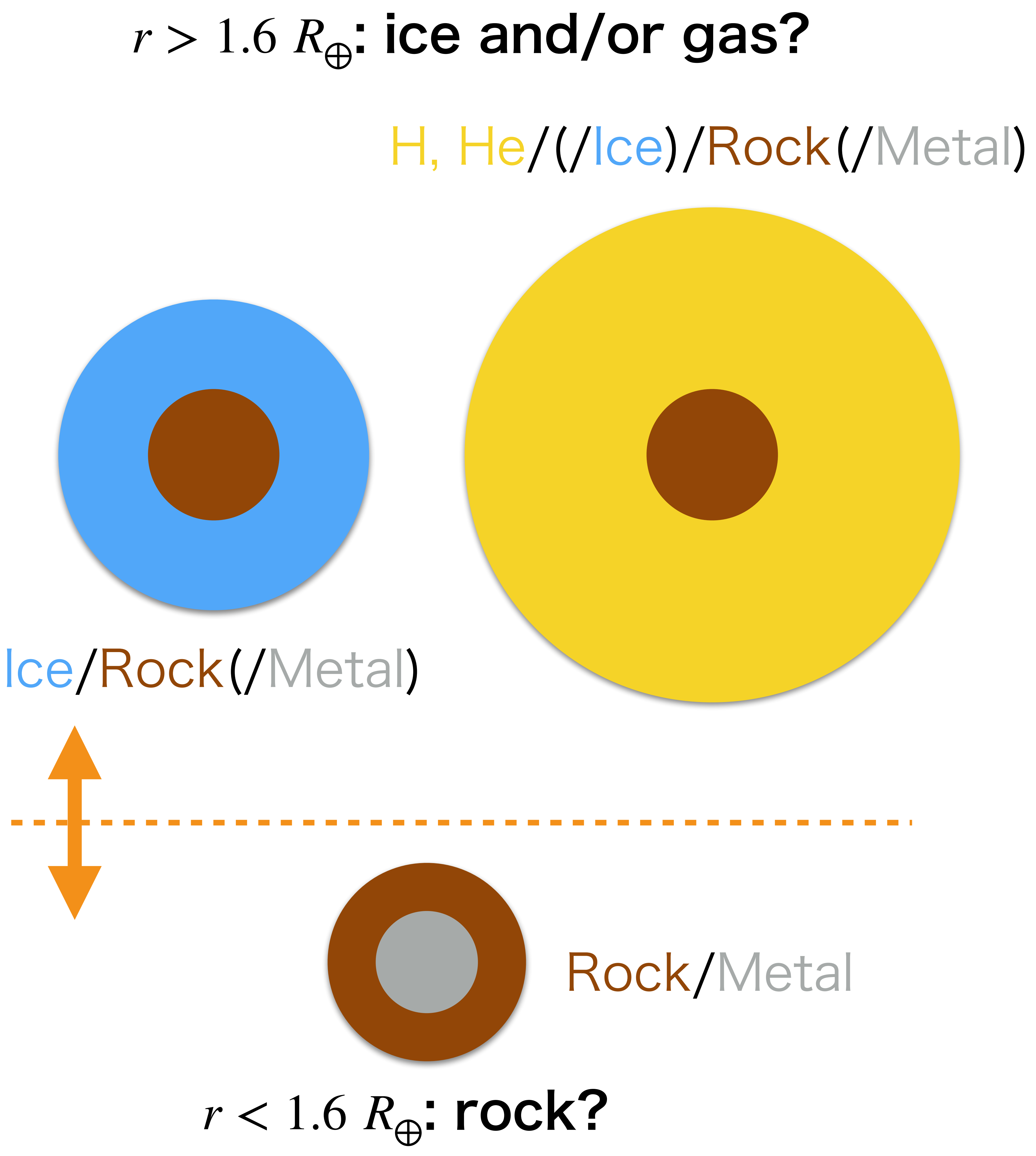


Tachinami et al. (2011) *Astrophys J.*

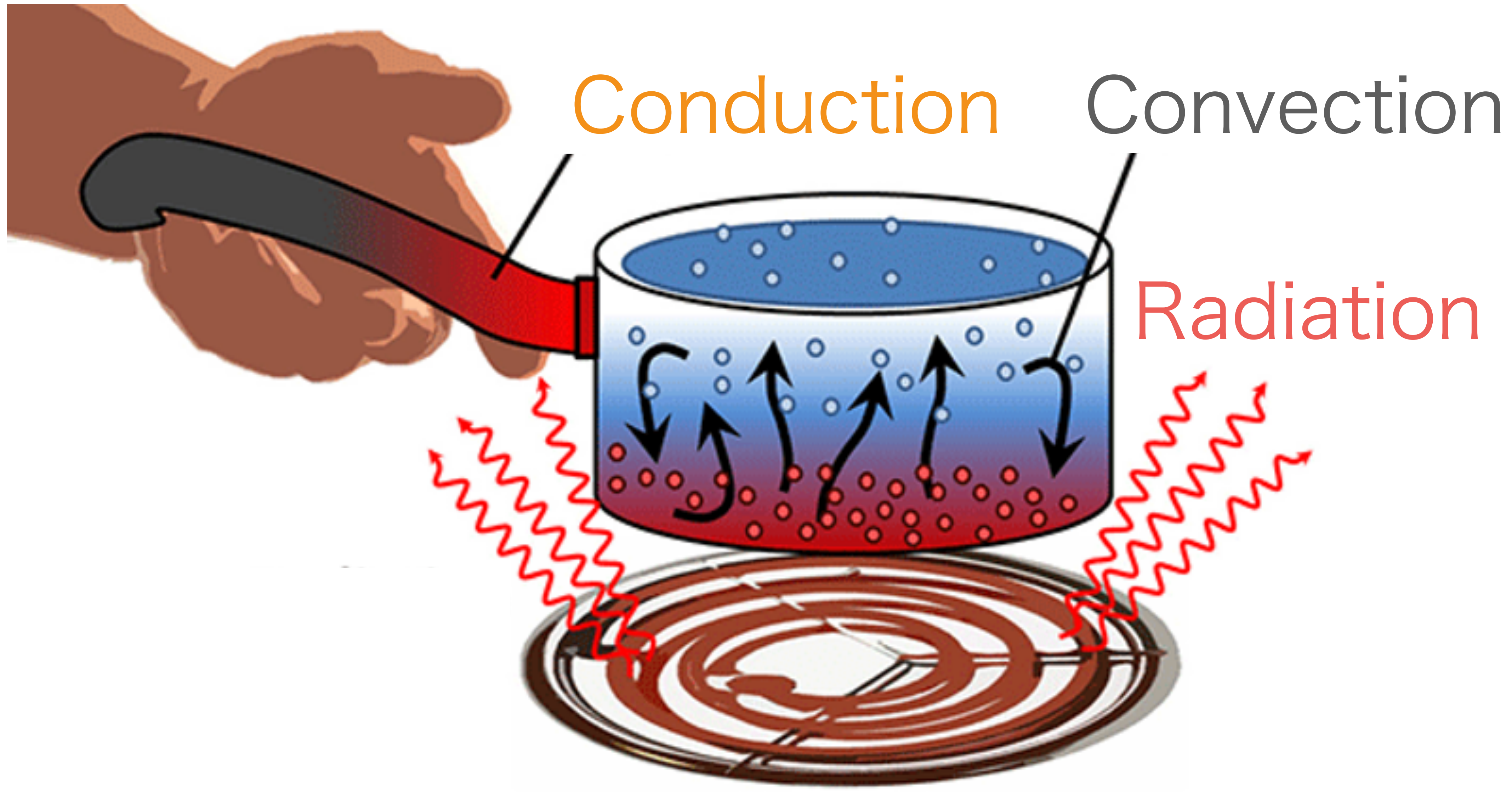
Exoplanet mass-radius relations

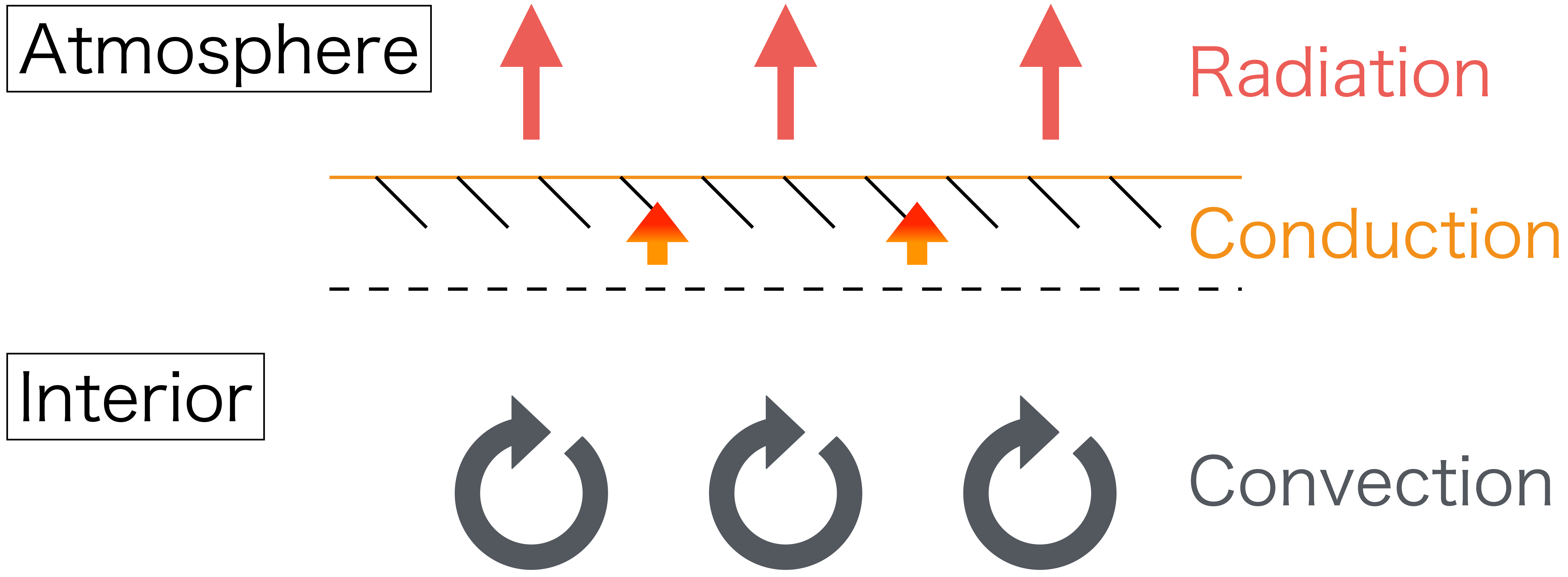


Zeng et al. (2019) *PNAS*



Energy transfer in the interior





- Energy transfer in the planetary interior: convection and conduction
- Because convection cannot operate at the material interface, a conductive layer develops

Conduction equation

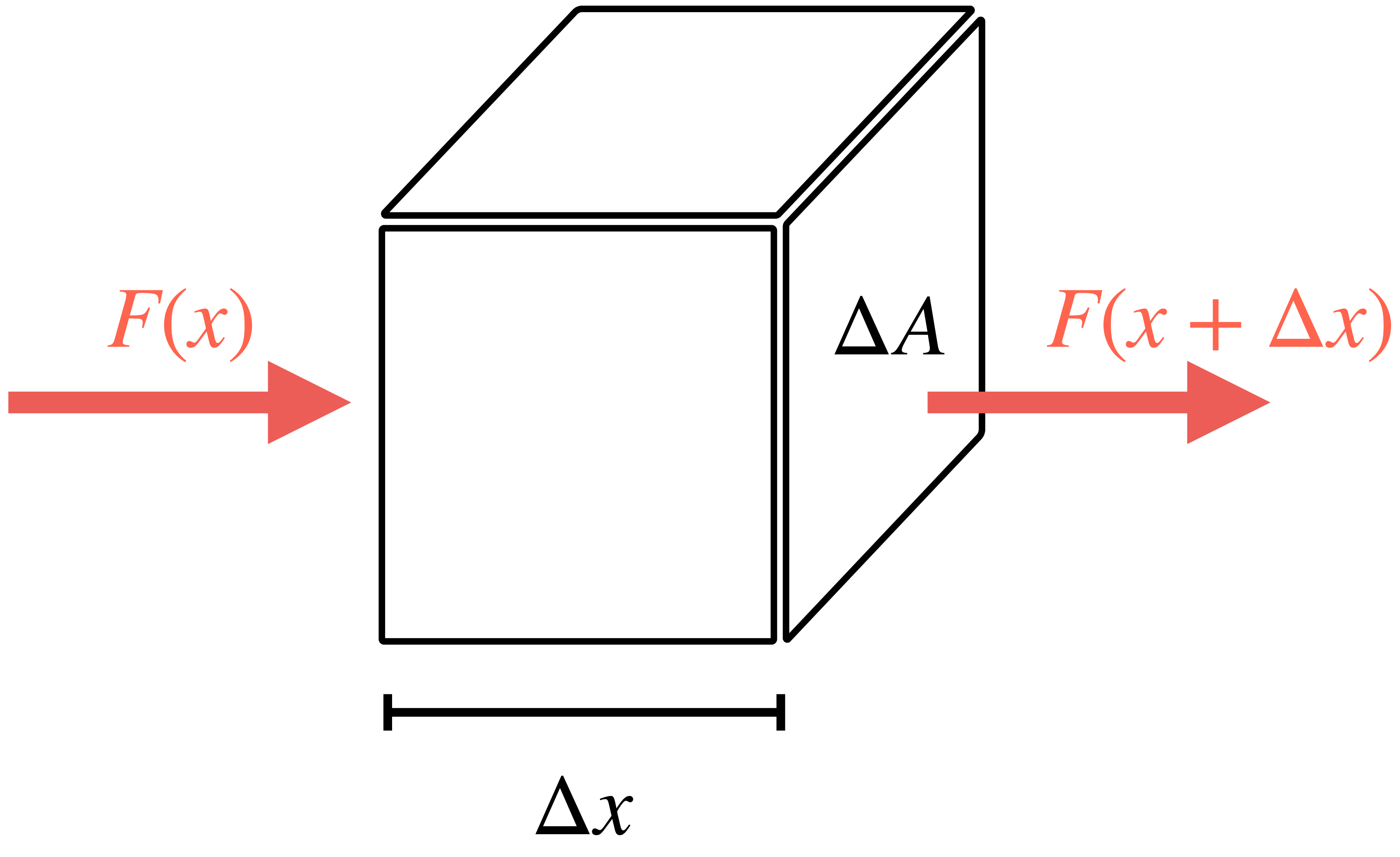
Conduction flux: $F = -k_{\mathrm{cond}} \dfrac{\partial T}{\partial x}$ — (1)

Conduction equation: $\rho c_p \dfrac{\partial T}{\partial t} = -\dfrac{\partial F}{\partial x}$ — (2)

where k_{cond}: thermal conductivity, c_p: heat capacity (per unit mass)

When k_{cond} is constant, Eq. 2 can be written as $\dfrac{\partial T}{\partial t} = \kappa \dfrac{\partial^2 T}{\partial x^2}$ — (3)

where $\kappa \equiv \dfrac{k_{\mathrm{cond}}}{\rho c_p}$ is the thermal diffusion coefficient



Timescale for thermal conduction

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{— (3)}$$

Thermal diffusivity of mantle rock $\kappa \sim 10^{-6}\ \mathrm{m^2\ s^{-1}}$ — (4)

The distance l and its conduction timescale τ has a relation, $\tau \sim \dfrac{l^2}{\kappa}$ — (5)

For the entire mantle $l \sim 3 \times 10^6\ \mathrm{m} \rightarrow \tau \sim 10^{11}\ \mathrm{year}$!

∴ Conductive cooling is inefficient for a planet

Rayleigh number and convective instability

- Whether a system starts to convect is determined by the Rayleigh number Ra,
- $Ra = \dfrac{\alpha \rho g \Delta T d^3}{\kappa \eta}$ = buoyancy/(heat conduction・viscosity) — (1)

  α: thermal expansion coefficient, ρ: density, g: gravitational acceleration,

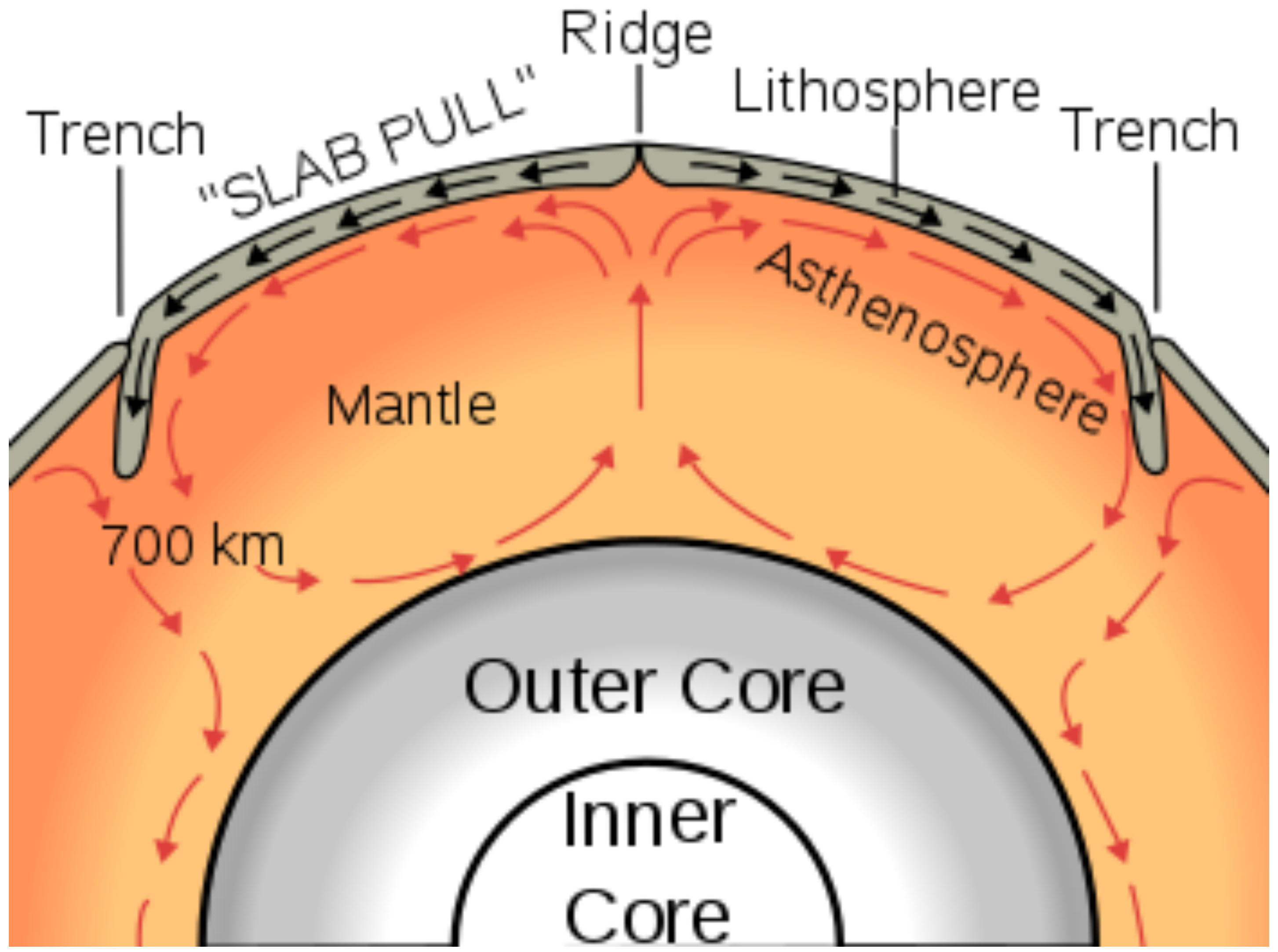
  ΔT: temperature difference between the top and the bottom,

  d: distance from the top to the bottom, κ: thermal diffusion coefficient,

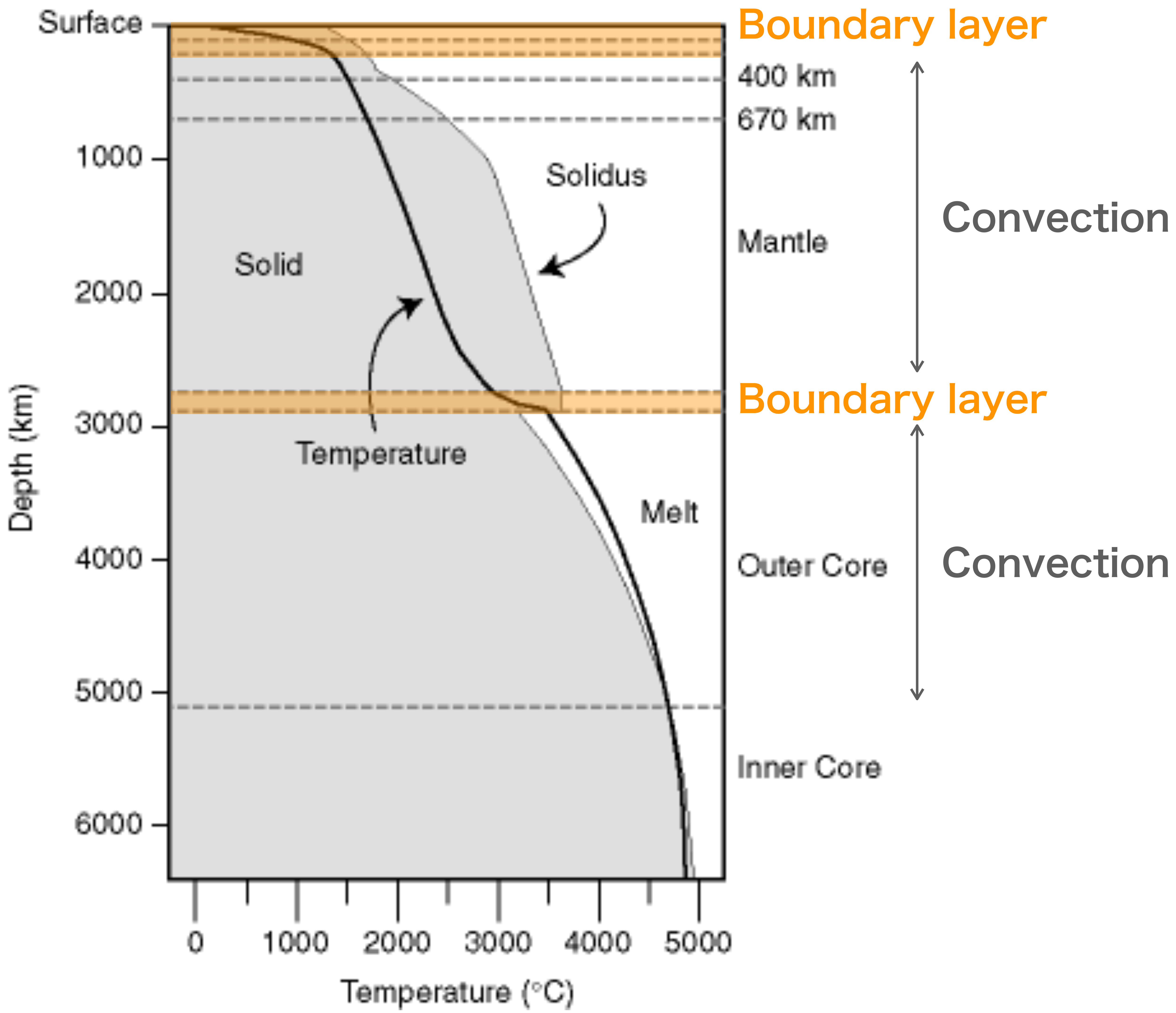
  η: viscosity coefficient

- Criterion for instability: $Ra \gtrsim 10^3$

  ↔ Earth's mantle $Ra \sim 10^7 - 10^8$



Temperature profile of Earth's interior



http://eqseis.geosc.psu.edu/~cammon/HTML/Classes/IntroQuakes/Notes/earth_origin_lecture.html

- Boundary layer profile:
  Temperature gradient to conduct energy

$$\frac{dT}{dr} \simeq \left(\frac{dT}{dr}\right)_{\rm cond} \equiv -\frac{F_{\rm int}}{k_{\rm cond}} \quad \text{— (1)}$$

  Thermal conductivity: $k_{\rm cond} \equiv \rho C_p \kappa$

- Convective layer profile:
  Adiabatic lapse rate

$$\frac{dT}{dr} \simeq \left(\frac{dT}{dr}\right)_{\rm ad} = -\frac{\alpha g T}{C_p} \quad \text{— (2)}$$

Derivation of adiabatic lapse rate (for physics students)

Entropy change: $dS = \left(\frac{\partial S}{\partial T}\right)_p dT + \left(\frac{\partial S}{\partial p}\right)_T dp = \frac{C_p}{T}dT + \left(\frac{\partial S}{\partial p}\right)_T dp$ — (1)

Gibbs free energy change $dG = d(U + pV - TS) = Vdp - SdT$ — (2),

$S = -\left(\frac{\partial G}{\partial T}\right)_p$, $V = \left(\frac{\partial G}{\partial p}\right)_T$ — (3).

$\therefore \left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial}{\partial p}\left(\frac{\partial G}{\partial T}\right)_p\right)_T = -\left(\frac{\partial}{\partial T}\left(\frac{\partial G}{\partial p}\right)_T\right)_p = -\left(\frac{\partial V}{\partial T}\right)_p$ — (4). (Maxwell relations)

Substituting Eq. 4 into Eq. 1, we obtain $dS = \frac{C_p}{T}dT - \left(\frac{\partial V}{\partial T}\right)_p dp$ — (5)

Finally, substituting $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$ — (6) into Eq. 5, we obtain $dS = \frac{C_p}{T}dT - \alpha V dp$. — (7)

$\therefore \left(\frac{\partial T}{\partial p}\right)_S = \frac{\alpha T}{\rho C_p}$ — (8) → $\left(\frac{dT}{dz}\right)_{\mathrm{ad}} = \frac{dp}{dz}\left(\frac{\partial T}{\partial p}\right)_S = -\frac{\alpha g T}{C_p}$ — (9)

Summary

- How to know planetary interior structures:
  - bulk density, seismology, gravity measurements, etc.
- Earth's interior: crust, mantle, outer and inner core
- Diversity in planetary compositions: rocky, gaseous, icy
- Planetary structure equations:
  - Hydrostatic equilibrium
  - Mass conservation
  - Equation of state
  - Energy transfer: conduction, convection, radiation

Report assignment

Summarize your answers into a short report and submit it by the beginning of the next lecture (either directly, to my post-box, or by e-mail to hiro.kurokawa@elsi.jp).

1. Thermal conduction determines the temperature profile in the boundary layer. Using the physical quantities given below, estimate the temperature gradient in the upper boundary layer of Earth's interior (the top ~100 km). Answer with one significant digit.

   $F_{\mathrm{int}} = 0.09\ \mathrm{W \cdot m^{-2}}$, $\kappa \simeq 1 \times 10^{-6}\ \mathrm{m^2 \cdot s^{-1}}$, $\rho \simeq 3 \times 10^3\ \mathrm{kg \cdot m^{-3}}$, $c_p \simeq 1 \times 10^3\ \mathrm{J \cdot kg^{-1} \cdot K^{-1}}$

   $$\therefore \left(\frac{dT}{dz}\right)_{\mathrm{cond}} = -\frac{F_{\mathrm{int}}}{k_{\mathrm{cond}}} = -\frac{F_{\mathrm{int}}}{\rho C_p \kappa} \simeq -\boxed{}\ \mathrm{K \cdot km^{-1}}$$

2. Let's assume that you are a hot-spring (*onsen*) enthusiast and want to dig for a hot spring of your own. Using the result of Q1, discuss how deep you need to dig a hole in the ground.